Visualization of Mathematical Concepts Using Innovative Technologies in Education

By

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2014-NUST-MS-ITE-63041

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A thesis submitted in partial fulfillment of the requirements for the degree of

Masters of Science in Innovative Technologies in Education (MS ITE)

In

School of Electrical Engineering and Computer Science,

National University of Sciences and Technology (NUST),

Islamabad, Pakistan.

(August, 2015)

Approval

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Dedication

Dedicated to my parents for their unconditional love, prayers and support throughout my life. Words can never be enough to describe how much I appreciate everything that both of you have done for me; my grandmother, for instilling her passion of education in all her grandchildren and being an excellent example of a woman of substance; my sister for being my best friend, confidante and guide; my extended family, cousins and friends for being my support system.

To all my teachers, from kindergarten to graduate level, who put in an effort to teach me, train me and inculcate a sense of duty and responsibility in me. To my spiritual parents: Madam Iram Anjum for teaching me never to compromise on my principles and values; Madam Robina Akram for showing me how to follow the right way; Sir Saleem Iftekhar for guiding me when I was lost; Sir Maajid Maqbool for believing in me when I was at my lowest; Dr. Tauseef Tauqeer for being kind in a very unkind world; Dr. Salma Siddiqui for being an excellent role model; Sir Ali Haider for showing me the importance of tapping into the universal mind; last but definitely not least, Dr. Ashraf Iqbal for being a magical mentor.

Certificate of Originality

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by any other person, nor material which to a substantial extent has been accepted for the award of any degree or diploma at NUST SEECS or at any other educational institute, except where due acknowledgement has been made in the thesis. Any contribution made to the research by others, with whom I have worked at NUST SEECS or elsewhere, is explicitly acknowledged in the thesis.

I also declare that the intellectual content of this thesis is the product of my own work, except for the assistance from others in the project's design and conception or in style, presentation and linguistics which has been acknowledged.

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Acknowledgement

First and foremost, gratitude to Allah Almighty for blessing me much more than I deserve.

Secondly, I am pleased to express my profound gratitude to my supervisor, Dr. Sohail Iqbal who has been an excellent source of guidance, knowledge and inspiration. I owe the timely completion of my thesis project solely to him, as despite his busy schedule he promptly replied to all emails and was available for answering all queries. His boundless knowledge on every aspect of the research study was very beneficial for me. I also want to profusely thank him for his role during the implementation phase of the project, the research findings could not have been so detailed and valid without his active involvement. I have learnt so much from him since the start of the project that I could not have wished for a better supervisor.

I am very grateful to Ms. Manzil Maqsood who has been my inspiration since before the start of the thesis project and has always been willing to share not only her specialized knowledge of education research, but also her vast network of contacts, most of which were acquired during her time in Oxford University UK. I am extremely lucky to have a Co-Supervisor who never compromises on quality and has been a great source of motivation throughout the thesis project.

I also want to thank the Committee members Dr. Muddassir Malik and Sir Jaudat Mamoon for their support, encouragement and cooperation.

Last, but not least, I want to express my gratitude to Sir Moin ud Din, Director Basic Sciences, SEECS NUST and Madam Atifa Kanwal, for their guidance and cooperation in the survey design and implementation phase, without which the experimental design could not be set up.

Abstract

Mathematics is one of the most important sciences, vital to and either directly or indirectly applicable in all aspects of life and all professions. It is also viewed by most students as a challenging and hard subject. High school mathematics is of paramount importance, especially for those students who will take up mathematics at the undergraduate and graduate level. Conceptual ambiguities and confusions in high school mathematics lead to poor performance at higher levels. The lack of understanding of the practical applicability of the subject in real life causes dissatisfaction and boredom in students. A possible solution to tackle the abstract and perplexing nature of mathematics is to use visualization (videos and animations) as a teaching aid. This study investigates the effect of using a specially designed visualization on the mathematical proficiency of high school calculus students.

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List of Abbreviations

NCTM- National Council of Teachers of Mathematics

NAS- National Academy of Sciences

RFT- Randomized Field Trials

PISA- Programme for International Student Assessment

TIMSS- Trends in International Mathematics and Science Study

CTGV- Cognition and Technology Group Vanderbilt

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1. Introduction

1.1. Background

Vision or sight is one of the five senses and a mode of receiving information from the outside world. Visualization is the process of creating visual content, through the use of pictures, diagrams, illustrations, concept maps, animations and video. It is a very strong means of communication and a way to effectively represent data in an organized manner. It is particularly useful when representing relationships or connections between two or more ideas. The use of visualization makes figurative and intangible concepts concrete (Arcavi, 2003, revised 2006). Therefore, visualization is a very powerful tool and teaching aid when used for education purposes.

Through the years, the definition and what comprises of as mathematical skill has evolved. In recent years, the focus has shifted from procedural drill or the act of solving a mathematical question using computations to a deeper, more meaningful and conceptual understanding of the subject as a whole and awareness of its usefulness and applicability in real world situations. (J Kilpatrick, 2001) has presented five interwoven and interdependent strands of mathematical proficiency namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Conventional teaching practices focus on drilling the set procedures/algorithm that when implemented, provides the solution to a mathematical question. Consequently, deep understanding and conceptual clarity is not achieved.

Calculus is one of the most important courses of mathematics at the advanced level classes (Ayub, Sembok, & Luan) and it poses a great challenge to teachers and students as it is an arcane and abstract subject (Gordon, 2004).

. The use of technology can transform a classroom from a dull and monotonous place to an exciting learning setting for mathematics. However, there is a scarcity of research on the effectiveness of the various technologies when used in educational settings.

1.2. Motivation

Currently the state of education in Pakistan is dismal. A low literacy rate of 55% ranks it as 160th among all the countries of the world (Archivist Online Blog, 2015). Apart from illiteracy, Pakistan faces a myriad of problems: growing numbers of out of school children, lack of adequate civic facilities in schools, teacher absenteeism and poor quality of education are to name a few.

Various studies have been conducted to assess the competency of students in different grades. An analysis of these studies indicate that in the 5th grade, on average, only half the syllabus content is mastered (UNESCO, 2003). Studies indicate that students performed poorly on questions requiring comprehension and problem solving skills (Aijaz, 2001). In 1995, a study (Mirza & Hameed, 1995) to test the competency of students from grades 1-5 showed the lowest scores in mathematics.

A look at the Annual Status of Education Reports from the recent years shows a bleak picture for mathematics education as well. In 2010, percentage of students in grade 1 who could recognize one digit numbers was 35.9 and two digit numbers was 25.8; students able to perform subtraction of two digit numbers was 4, and three digit division could be done by 2.1% students. For grade 5, students who could recognize one digit numbers was 5.5 and two digit numbers was 21; students able to perform subtraction of two digit numbers was 36.5, and three

digit division could be done by 34.3% students. For grade 10, students who could recognize one digit numbers was 0.6 and two digit numbers was 3.7; students able to perform subtraction of two digit numbers was 11.1, and three digit division could be done by 83.8% students (ASER, 2010).

In 2011, 37.8% students of grade 1 could recognize one digit numbers and 25.5% could recognize two digit numbers; two digit subtraction could be done by 3.5%, and three digit division could be done by 1.5% students. For grade 5, students who could recognize one digit numbers was 5.8 and two digit numbers was 21.4; students able to perform subtraction of two digit numbers was 32.4, and three digit division could be done by 37.3% students. For grade 10, students who could recognize one digit numbers was 1.3 and two digit numbers was 4.1; students able to perform subtraction of two digit numbers was 9.5, and three digit division could be done by 84.1% students. (ASER, 2011)

In 2012, percentage of students in grade 1 who could recognize one digit numbers was 33 and two digit numbers was 29.6; students able to perform subtraction of two digit numbers was 4.2, and three digit division could be done by 3.2% students. For grade 5, students who could recognize one digit numbers was 3.3and two digit numbers was 18.7; students able to perform subtraction of two digit numbers was 29.9, and three digit division could be done by 43.8% students. For grade 10, students who could recognize one digit numbers was 0 and two digit numbers was 1.5; students able to perform subtraction of two digit numbers was 8.5, and three digit division could be done by 90% students. (ASER, 2012)

In 2013, percentage of students in grade 1 who could recognize one digit numbers was 35.1 and two digit numbers was 28.7; students able to perform subtraction of two digit numbers was 4.2, and three digit division could be done by 2.2% students. For grade 5, students who could recognize one digit numbers was 5 and two digit numbers was 18.2; students able to perform subtraction of two digit numbers was 30.5, and three digit division could be done by 43.2% students. For grade 10, students who could recognize one digit numbers was 3.2 and two digit

numbers was 4.9; students able to perform subtraction of two digit numbers was 9.1, and three digit division could be done by 80.1% students. (ASER, 2013)

The findings of these reports are tabulated in Table 1-1.

Table 1.1 ASER findings on Arithmetic Proficiency in Primary grades

Year	Grade	1 digit number recognition (%)	2 digit number recognition (%)	2 digit subtraction (%)	3 digit division (%)
2010	**1**	35.9	25.8	4.0	2.1
	5	5.5	21	36.5	34.3
	10	0.6	3.7	11.1	83.8
2011	**1**	37.8	25.5	3.5	1.5
	5	5.8	21.4	32.4	37.3
	10	1.3	4.1	9.5	84.1
2012	**1**	33.0	29.6	4.2	3.2
	5	3.3	18.7	29.9	43.8
	10	0.0	1.5	8.5	90.0
2013	**1**	35.1	28.7	4.2	2.2
	5	5.0	18.2	30.5	40.2
	10	3.2	4.9	9.1	80.1

There has been no significant improvement in the survey findings over the recent years, in fact there have been slight increases in each category. The consistently poor performance in mathematics was the main motivation behind this thesis project, as it was hoped that this

research study would prove to be a very small, but significant first step in the direction for a policy change at the national level.

1.3. Study Overview

The purpose of this study was to minimize the gaping hole in the research world by exploring the effectiveness of using technology as a teaching aid, along with regular teaching methodology, on the learning and proficiency of the students. It aimed at investigating the effects of using visualization along with regular teaching methods on the mathematical proficiency of Calculus. The focus was on two out of five strands of mathematical proficiency, namely conceptual understanding and adaptive reasoning (J Kilpatrick, 2001). High school calculus was targeted.

Therefore, the research problem was to inspect the impact of visualization on the conceptual understanding and adaptive reasoning of high school calculus students. A quasi-experimental study was conducted with the pre-test post-test control and experimental group design (n=33 each). A significant difference in the means of both the groups was revealed when quantitative data analysis was performed and a strong effect size was found. Therefore, it was concluded that visualization as a supplement to teaching significantly increases the conceptual understanding and adaptive reasoning of high school calculus.

The study had certain limitations as it dealt with a very small portion of high school calculus syllabus i.e. the topics of integration and differentiation. Moreover, due to time constraints the implementation phase was not very extensive. A longer study covering a larger portion of high school calculus syllabus would potentially further strengthen the findings.

2. Literature Review

There have been many attempts to define and understand mathematical skill and aptitude. The great mathematics educator Richard Skemp introduced the terms “Instrumental Understanding” and “Relational Understanding” for mathematics (Skemp, 1977), where Instrumental Understanding is explained to be a mechanical process of learning the method or algorithm of solving a mathematics problem and Relational Understanding is a more meaningful grasp of the structure of mathematics and relations. Skemp propagated the importance of Relational Understanding due to a higher adaptability to new tasks and ease of remembering, thus being more beneficial in the long term (Skemp, 1977).

In 1984, the world of mathematics education saw a standards based reform, or standards based change, put forward by National Council of Teachers of Mathematics (NCTM). The NCTM set standards for curriculum and evaluation (1989); for teaching (1995); principles and standards for school mathematics (2000) (Kilpatrick, 2011).

After the “cognitive revolution” (Gardner H. , 1985) the definition of proficiency broadened; instead of focusing only on knowledge, emphasis was laid on how well the student can apply this knowledge in the appropriate circumstances (Schoenfeld, 2007).

The late 80’s saw a Calculus Reform Movement in USA (Wilson, 1997). This was a result of widespread dissatisfaction and general discontent with the Calculus course all over the world. The Movement comprised of abundant investment in development and technology but a dearth of cognitive research (Tall, 1992).

2.1. Birth of "Mathematical Proficiency"

In 1998, the National Academy of Sciences (NAS), on the request of the US Department of Education and National Science Foundation (NSF) established a committee for research in mathematics education and learning. An eighteen month study was conducted to find recommendations for improving learning in mathematics. The output of the study was an extensive report titled "Adding It Up" and was released in 2001 (Kilpatrick, 2011).

The most important theme of Adding It Up was the attempt to define and characterize successful mathematical learning. Many terms were considered, including "mathematical literacy", "numeracy", "mastery", "competence", but none of these were able to grasp the full nature of mathematical learning. Finally the term "mathematical proficiency" was coined (Kilpatrick, 2011). According to the experts who devised this term, Mathematical Proficiency encompasses both skill and understanding of the subject, without preferring one over the other and transcends beyond that (Kilpatrick, 2011).

Mathematical Proficiency can be defined by five interwoven and interdependent strands that are to be developed concurrently. A pictorial depiction is shown in Figure 2.1.

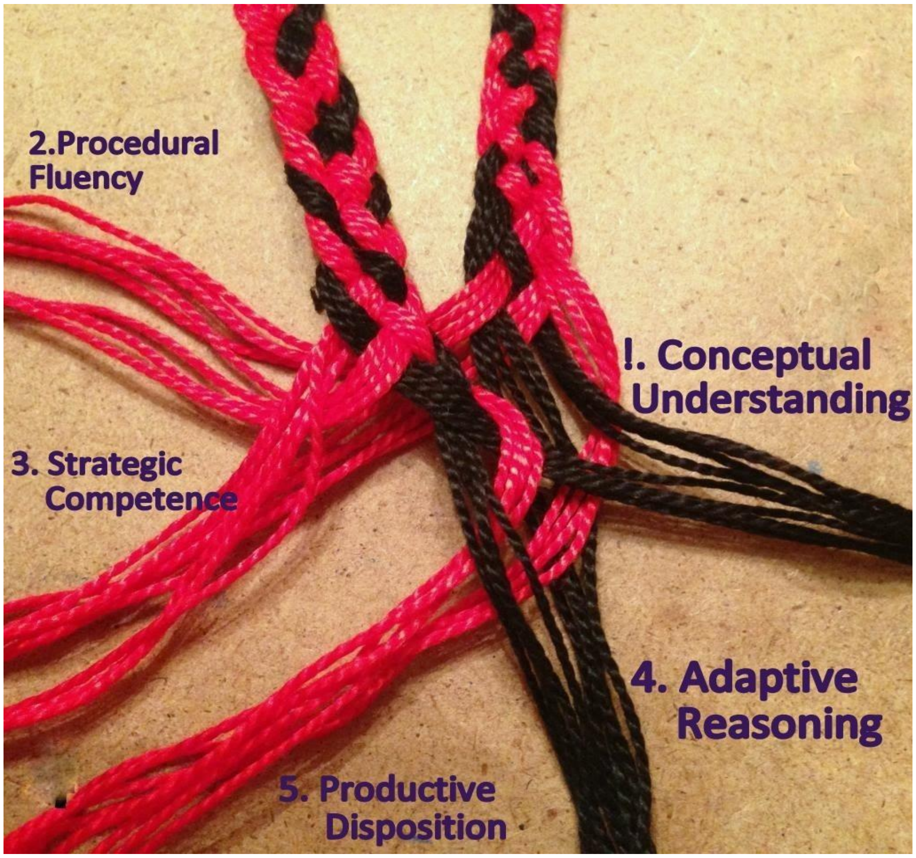


Figure 2.1 Five Strands of Mathematical Proficiency

Image adapted from (2015)

They are described below (Danley, 2008) (J Kilpatrick, 2001) (Groves, Developing Mathematical Proficiency, 2012) :

2.1.1. Conceptual Understanding

Conceptual Understanding refers to a firm comprehension of the rudimentary fundamentals of mathematics including the knowledge of what mathematical symbols and procedures mean. It requires more than computational skills or a bank of isolated and disconnected pieces of information. Through understanding, a meaningfully connected reservoir of facts and skills is formed that can be applied when solving unfamiliar problems.

Conceptual understanding denotes an insightful comprehension of mathematical concepts, operations, and relations. It is a cohesive and purposeful grasp over mathematical functions. Possessing sound conceptual understanding requires being aware of the importance of a mathematical idea and being able to identify which mathematical idea is applicable in a particular setting.

A person with a thorough command over the concepts will possess a structured and coherent knowledge base; have the ability to form meaningful connections between data and link new knowledge with their previous knowledge.

2.1.2. Procedural Fluency

Procedural Fluency basically refers to swiftness and precision in performing mathematical operations. It is the "knowledge of procedures… and skill in performing them flexibly, accurately, and efficiently". This includes fluent paper and pencil computations, as well as mental calculations.

For pencil and paper computations, standard procedures are applicable for all problems. However for mental calculations, the strategies are distinctive, depending on the numbers involved and the person's ability to perform the calculations.

Computational skills are integral for a wholesome and complete math experience. Similar to the experience of a student who is struggling with reading and has to focus so hard for word recognition that he loses sense of the paragraph on the whole, a student struggling with procedural fluency in math will be unable to grasp how well structured and organized the subject is. So a lack in this particular strand has drastic effects on the ability to master the other four strands as well.

2.1.3. Strategic Competence

Strategic competence is the skill of articulating mathematical problems, symbolizing them and solving them. This is similar to problem solving and problem formulation, terms that are commonly used in mathematics education and cognitive science.

Mathematical problem solving is at the core of mathematics learning. The spirit of problem solving is in accepting the challenge of undertaking a new, unacquainted task for which there is no obvious solution or an algorithm that can be directly plugged in.

Strategic Competence comes into play when a complex situation arises, unlike the sums likely to be found in a workbook, that pose a direct question and there exists a fixed sequence of steps that when employed can get the answer. In such a situation, the student has to identify:

- The information needed in order to solve the problem;
- The information that is irrelevant and needs to be disregarded.

The ability to gauge the situation, identify the problem, and formulate an efficient strategy to solve the identified problem is strategic competence.

Problem solving overlaps with the fifth strand of mathematical proficiency i.e. productive disposition, as it involves the belief that application of mathematical procedures is the solution to the given problem.

2.1.4. Adaptive Reasoning

Adaptive Reasoning is called "*the glue that holds everything together*" (J Kilpatrick, 2001, p. 129). It refers to the rational and logical thought process about the relationships among mathematical concepts and situations. This thought process is meticulously formed by considering all alternatives, weighing all the options and the reasoning is accurate and usable. A person proficient in adaptive reasoning is able to reflect and justify his reasoning.

It is the ability to realize that there are several relationships between mathematical concepts and to make use of those to apply problem solving strategies, to find meaning in the algorithm (fixed procedural steps to get the answer to mathematical question).

Adaptive Reasoning is taking place when (ACARA, 2014):

- The thought process and choices are being explained;
- when the strategies are being inferred;
- when conclusions are justified;
- when associated ideas are compared and contrasted.

Initially, the term “logical reasoning” was considered to define the fourth strand. However, it is an inadequate term as other types of reasoning are very important for complete mathematical proficiency, including inductive, deductive and plausible reasoning.

The fourth strand is very closely linked to the first one i.e. conceptual understanding. One cannot exist without the other (Danley, 2008).

2.1.5. Productive Disposition

Productive disposition is related to the way an individual sees or regards mathematics. A person with a productive disposition views mathematics as logical, reasonable, subjective; as a subject that can be mastered and used in daily life, and with industrious effort it can be understood.

This can be done if a student is given ample opportunities to make sense of and understand mathematics; and realize the advantages of persistent and diligent efforts in the subject. This strand is a direct result of the previous four strands, a person who has developed the previous four is one who believes that mathematics is not an esoteric, abstract field; instead it is practical and tangible, thus possesses productive disposition. The converse is also true: a

person with a positive attitude towards mathematics is more likely to gain command over the previous four strands.

As productive disposition is about a certain approach and outlook towards mathematics, therefore there was some debate among the NAS team members about including it as a strand of proficiency. However, these concerns were put to rest by the teachers on the committee who propagated that proficiency must include an affective component because it cannot be achieved if students are disinclined and averse to mathematics in general.

2.1.6. Summary of Five Strands

The five strands are intertwined and mutually supportive of each other; enhancement of one strand fosters the development of the others. Command on all five results in an enriched knowledge base and enables the student to use the learned skills flexibly and effectively in unfamiliar situations.

Table 2-1 summarizes the five strands of mathematical proficiency.

Table 2.1 Five Strands Summarized

STRANDS	**MEANING**
Conceptual Understanding	Command over math concepts, functions, symbols and their relationships.
Procedural Fluency	Ability to perform math procedures adaptably, precisely, capably, and aptly.

Strategic Competence	Skill to frame, symbolize, and provide a solution to mathematical problems efficiently
Adaptive Reasoning	Ability reflect, reason (logically, inductively, deductively, plausibly) , explain and justify decisions
Productive Disposition	Predisposition to view mathematics as practical, valuable, beneficial; in addition to self-belief in ones own ability to be able to master mathematics with hard work and determination

2.2. More on Mathematical Proficiency

In 2007, Schoenfeld (Schoenfeld, 2007) identified the qualities that a capable mathematician must have:

- find results and solutions;
- add on to known results;
- apply known results in new circumstances;
- possess a mathematical disposition-to take on complex mathematical challenges;
- a firm resolve to keep trying long after others give up.

(Schoenfeld, 2007) further states that mathematical expertise comprises of the following:

2.2.1. Knowledge Base

There have been many debates on setting the curriculum, what to include at what stage of education, which topics to cover in detail and which are to be touched upon only. There is no absolute answer to this and experts differ on opinion.

Schoenfeld states that having a huge knowledge base, both in breadth and depth is imperative for mathematical proficiency. However, assessment has an even bigger role to play in this case as only through assessment the gaps in knowledge can be identified.

Moreover, the design of the assessment plays a crucial role in accurately determining the proficiency of the student. Therefore, assessment must be aimed at measuring skills, conceptual clarity and problem solving skills.

2.2.2. Strategies

(Polya, 1957) describes a list of heuristic strategies to be followed in order to solve a mathematical problem:

i. think and identify the relevant procedure or mathematical operation;

ii. generate a similar, easier related problem;

iii. solve the related problem;

iv. figure out how to exploit the solution to solve the original problem.

2.2.3. Meta Cognition

Meta cognition is knowing when and how to use mathematical knowledge effectively. It includes monitoring, evaluation and self-regulation: reflecting on the progress during the problem solving process and incorporating this feedback in the process to alter the plan when and if needed.

Figures 2.2 and 2.3 show a contrast between an attempt at problem solving with and without meta cognition. Images adapted from (Schoenfeld, 1991).

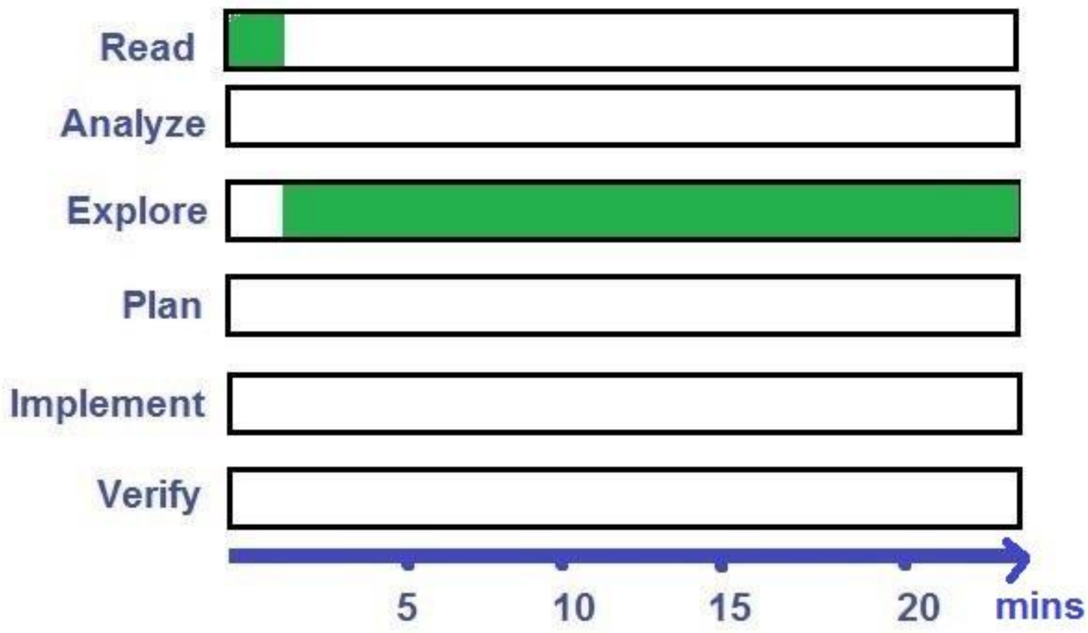


Figure 2.2 Problem Solving Without Meta Cognition

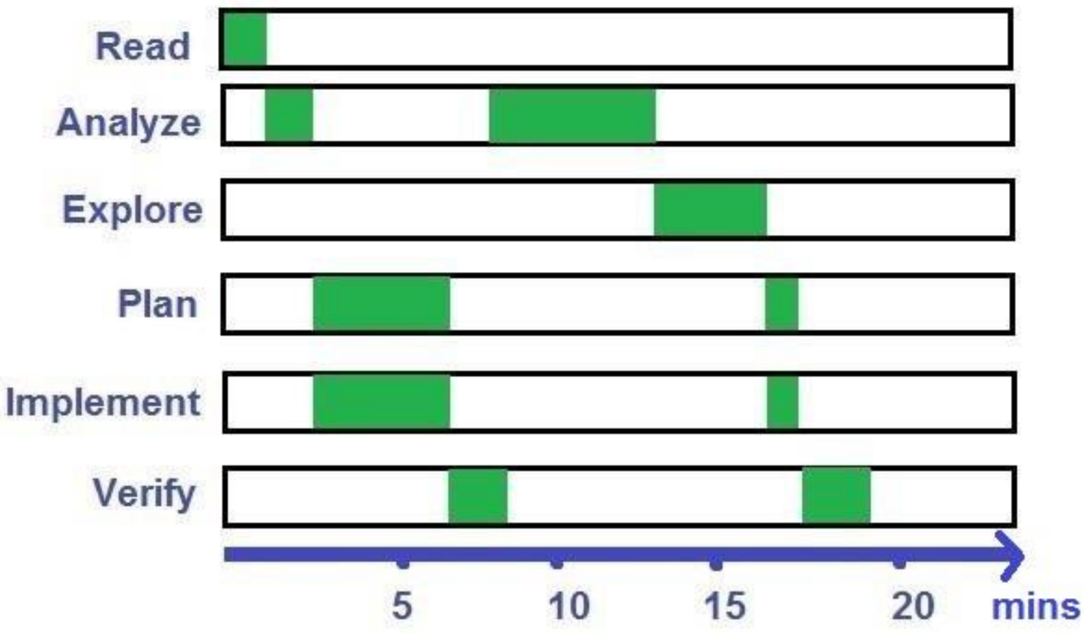


Figure 2.3 Problem Solving With Meta Cognition

2.2.4. Beliefs and Disposition

A mathematically proficient person believes that math is supposed to make sense, mathematical operations and symbols are not meaningless, and working them will produce sensible solutions to problems.

2.3. Educational Policy for Proficiency- Where the world stands

Despite being formulated in 2001, there are very few countries with an educational policy that focuses on all the five strands of mathematical proficiency. Two countries, Singapore and Australia, have emerged as the prominent names in this regard. Their policy and its impact are explained below.

2.3.1. Singapore

In 2006, the focus of mathematics teaching and learning was shifted from attainment of skills to growth and enhancement of a person's intellectual aptitude (Dave, 2012). The present framework for mathematics education in Singapore, titled Singapore Math, is similar to the five strands of mathematical proficiency defined by the National Research Council USA (J Kilpatrick, 2001) and the qualities identified by (Schoenfeld, 2007) ; and is one of the most comprehensive and inclusive (Dave, 2012).

It is centered on problem solving and comprises of five pillars, namely:

1. **Concepts**
    a. Numerical
    b. Algebraic
    c. Geometrical
    d. Statistical
    e. Probabilistic
    f. Analytical
2. **Processes**
    a. Reasoning, Communication and Connection
    b. Thinking Skills and Heuristics
    c. Applications and Modelling

3. **Skills**
    a. Numerical Calculation
    b. Algebraic Manipulation
    c. Spatial Visualization
    d. Data Analysis
    e. Measurement
    f. Use of mathematical tools
    g. Estimation
4. **Attitudes**
    a. Beliefs
    b. Interest
    c. Appreciation
    d. Confidence
    e. Perseverance
5. **Metacognition**
    a. Monitoring one’s own thinking
    b. Self-regulation of learning

2.3.1.2. Assessments and Rankings

The overall education system of Singapore is acknowledged as one of the best performing in the world (Mourshed, Chijioke, & Barber, 2010) and according to the Global Competitiveness Report 2013-2014, it is ranked among the top in “Quality of Math and Science Education” (Schwab & Sala-i-Martín, 2013).

The excellence in policy and methodology for mathematics education and learning is clearly reflected in the students’ performance in comparison to international standards. Singapore had the second highest rank in PISA 2012 for Mathematics with a mean score of 573 (PISA, 2013). Since 1995, Singapore has been in the top three positions for Mathematics (Grades 4 and 8) among the world. Inculcating the five strands of mathematical proficiency (J Kilpatrick, 2001) is showing excellent results.

2.3.2. Australia

In Australia, after slight modifications, the first four strands have been implemented since 2013, namely Understanding, Fluency, Problem Solving, and Reasoning (Groves, 2012). There has already been a reported change in the students' attitudes as they are able to explain and justify their thinking, choices, and strategies (ACARA, 2014).

2.4. Identifying Gaps

Researchers and mathematics instructors widely acknowledge that mathematics, especially calculus concepts are arcane and abstract, thus teachers have difficulty explaining them and students have trouble grasping them (Gordon, 2004). Many students are unable to form conceptual associations between mathematical concepts and relations, have trouble comprehending representations of these concepts, and cannot make the link between representation and abstractions (Hasselbring, Lott, & Zydney, 2006). (Hasselbring, et al., 1991) after a 2 year study concluded that often students are unable to use their previously acquired knowledge extemporaneously to solve problems unless they are directly informed about the applicability of the relevant piece of knowledge in that particular problem.

(Lesh, 1981) stated that the mere teaching of an idea or mathematical concept does not necessarily result in the student being able to form linkages with his previous knowledge, and thus, may not be able to identify the situation in which that particular concept is applicable. Without these linkages and the ability to conceptually apply the mathematical knowledge to real world situations, the knowledge gained is not useful.

Latest research indicates a need for reforms in Calculus education at the high school level. *''[Calculus] is both a climax of school mathematics and a gateway to further theoretical developments''.* The present methodologies of teaching Calculus at the school level focus on the procedure instead of structural understanding (Hoffkamp, 2011). This poses a possible

threat for students who take up Calculus at the college level of developing a predisposition to focus on the technique oriented, procedural aspects (Ferrini-Mundi, 1992). Moreover, research focused on conceptual understanding of Calculus indicates the inability of current teaching methods in helping students overcome learning difficulties (Orton, 1986).

(Hasselbring, Lott, & Zydney, 2006) identify the need for research in the use of technology for innovative approaches to mathematics education. Recent developments in technology have provided the world with substantial amount of relatively low cost technological tools and softwares that can be used as teaching aids in the classrooms. However, there is a severe deficiency of research on their effectiveness and identification of best practices and pedagogical techniques that can be embedded in technology for effective math instruction.

2.5. Visualization- A Possible Solution?

Vision is the most important input sensory mechanism (Adams R.D & Victor, 1993) and the human perception is strongly visual (Guzman, 2002). Mathematicians and scientists have always used their “mind’s eye” to envision the abstract items and procedures involved in mathematics (Palais, 1999). Mental visualization or imagery is a strong tool for in depth understanding of any concept. The ability to mentally visualize or “see” a concept is a synonym for conceptual understanding (Goldstein & Bloomfield, 2013). Recent developments in technology have provided us with tools to express these abstract pictures in the form of precise, tangible and objective visualizations (Palais, 1999). Mathematical concepts and relationships can be intuitively depicted in a variety of ways. The use of these depictions is very valuable in mathematics education as they can communicate mathematical theory in a versatile manner, which would otherwise be difficult to explain analytically and/or logically. (Guzman, 2002).

Renowned mathematics education researcher David Tall establishes the need for visualizations in mathematics in his book (Tall, 1990) by stating that students should be encouraged to construct their own concept images. This can be done by providing rich learning experiences to give better intuitions. He rules out language as incapable of achieving this on its

own and stresses the need for alternative forms of communication i.e. visualizations and animations, which can provide an opportunity to gain deeper insight and understanding. (Tall & Sheath, 1983) further states the limitations of illustrations, pictures and diagrams in books. He identifies two major problems in books:

- The images are static, therefore cannot deliver and accurately convey the dynamic nature of mathematical concepts;
- The images are limited in variety, thus present a myopic view of the concepts.

Moreover, (Tall, 1992) identified the inadequacy of graphical displays and stressed the need for movement between representations, which can be done through visualization techniques.

Previous researches have established the positive effects of visualization. (Hoffkamp, 2011) states that the interactive visualizations activate the intuitive and logical approach to concepts of calculus. Through visualization a wide angled view of concepts can be provided and the various ways through which they can be realized are easily depicted, therefore stronger intuitive sense can be developed (Tall, 1991). (Arcavi, 2003, revised 2006) claims that often while solving a mathematical question, there is a gap/mismatch between what the student was expecting and what the correct result is; visualization can help the student bridge this gap. (Arcavi, 2003, revised 2006) also states that visualization guides the analytical development of a solution as it meaningfully organizes the data.

Visual embodiment of mathematical concepts is a great help for those scientists who need to use mathematics regularly in their work, but are not expertly proficient in the subject and find it hard to master the abstract notations and formulae (Palais, 1999).

2.6. Previous research on visualization in mathematics education

The Jasper Project (Cognition and Technology Group at Vanderbilt, 1990)**:**

Cognition and Technology Group Vanderbilt (CTGV) produced a series of 12 videos that target the middle and high school level mathematical problem solving named the Jasper project. The series comprises of characters around whom a story close to real life revolves. The characters are faced with a challenge and the viewer is required to solve this challenge using his previously acquired mathematical knowledge and the information provided in the video.

A comprehensive research study was conducted in sixteen schools spanning over nine states of USA to evaluate the effects of Jasper on the learning and performance of students. Classes were classified into two groups: control group that was not exposed to the Jasper series and experimental group that used three to four videos from the Jasper series over the entire academic year. Pre-test scores for both the control and experimental groups were the same.

Post test data indicated that despite the experimental group classes spending less time on the original mathematics curriculum, the students scored the same or higher on standardized tests; performed better on word problems; had a more positive approach towards mathematics and saw it as useful and applicable in real life.

3. Research Framework

3.1. Types of Research in Education

Research is the implementation of a set of methodical, logical and replicable procedures with the intent of providing dependable knowledge. This is done by identifying, observing and analyzing the relationships between different variables, wherein a variable is any characteristic that is not fixed and is of some consequence to the researcher. Knowledge gained through research is not irrefutable or absolute, instead it is comparative and relative (Wolf, 2005).

There are various types of educational research methodologies. They may be classified in the following ways:

- **By topic**: The studies are grouped according to the particular phenomena under investigation.
- **By the purpose of investigation**:
    - **Exploratory:** when the study is conducted to explore the theoretical understanding about the phenomena under analysis. The output is produced by analyzing the relevant variables, their relationships and effects on the dependent variable.
    - **Confirmatory:** when the study is conducted to test an existing theoretical model that is generated by previous research findings and observations. The output is produced by gathering and analyzing field data.
- **By the nature of information**:
    - Historical: deal with data about past events and utilize primary and secondary sources of information.
    - Survey: deal with data about an individual or a group at a specific point in time.

    - Longitudinal: deal with data collected over a long period of time.
    - Case: deal with detailed data about an individual or a group.
    - Experimental: deal with data collected as a result of certain interventions done on an individual or a group.
        - Quasi-experimental: is considered a sub group of experimental studies and is a cross between experimental and survey studies. Partial (lesser than in Experimental) control is exerted over the research situation. Fairly strong causal conclusions can be drawn from the data. (More details in the coming section).
- **By type of data collected**:
    - Quantitative: Quantitative research methods were enable researchers to study the natural and other sciences. They are now employed in social sciences and educational researches too. It includes collecting, synthesizing and analyzing quantitative data (Hohmann, 2005).
    - Qualitative research methods were originally used to study social and cultural phenomenon. The types of qualitative methods include action research, case study, and ethnography (Hohmann, 2005).

Qualitative and quantitative studies are not in opposition to one another, in fact they can be complementary and when implemented together, could result in a stronger research study (Wolf, 2005)**.**

Figure 3.1 summarizes the types of education research methodologies.

Research Framework

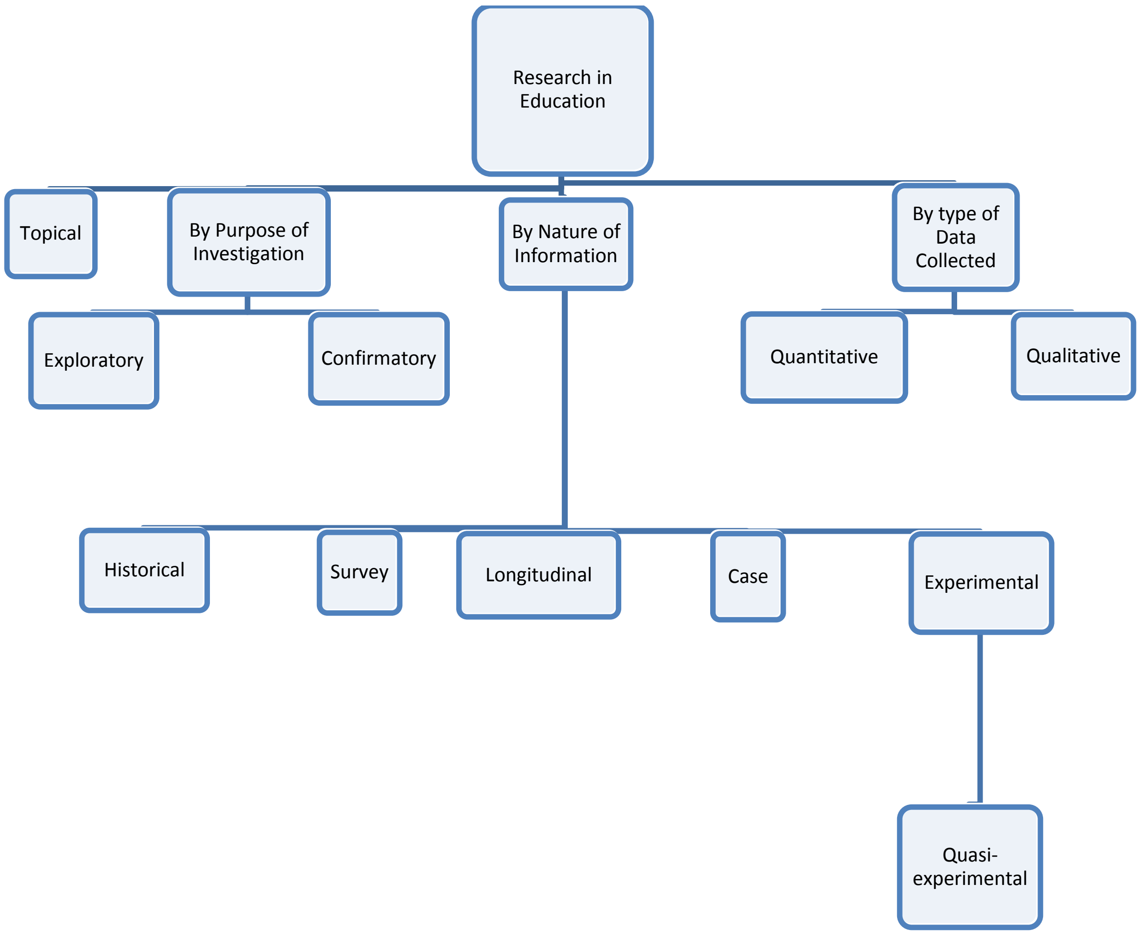


Figure 3.1 Classification of Educational Research Methodologies

3.2. Experimental Research

The word "experiment" is defined as an investigation under measured and controlled conditions; the aim of such an investigation is to exhibit an acknowledged truth, to discover, to confirm or reject a hypothesis, or to explore the usefulness of an untested phenomenon (Dictionary.com, 2015).

By their very nature, humans experiment with different ways of doing things (Shadish, Cook, & Campbell, 2002).

The basic feature common to all experimental researches is the deliberate variation of a factor (variable) to discover and observe any resulting changes in some other factor (variable) (Shadish, Cook, & Campbell, 2002).

An experimental study includes the following features: (Cohen, Manion, & Morrison, 2007)

- control group(s)
- experimental group(s)
- random allotment to control and experimental group(s)
- pre-test of both groups to confirm equivalence
- intervention(s) to the experimental group(s)
- isolation, control and manipulation of independent variables
- post-test of both groups to observe the resulting changes (if any) on the dependent variable
- non contamination between the control and experimental groups.

Experimental Studies can take the following forms (Cohen, Manion, & Morrison, 2007):

- one control and experimental group, pre-test post-test design
- two control groups and one experimental group, pre-test post-test design

- one control and experimental group, post-test design
- two experimental groups, post-test design
- two treatment, pre-test post-test design
- Solomon four group design
- matched pairs design
- parametric design
- factorial design
- repeated measures design

Figure 3.2 shows the types of Experimental Studies.

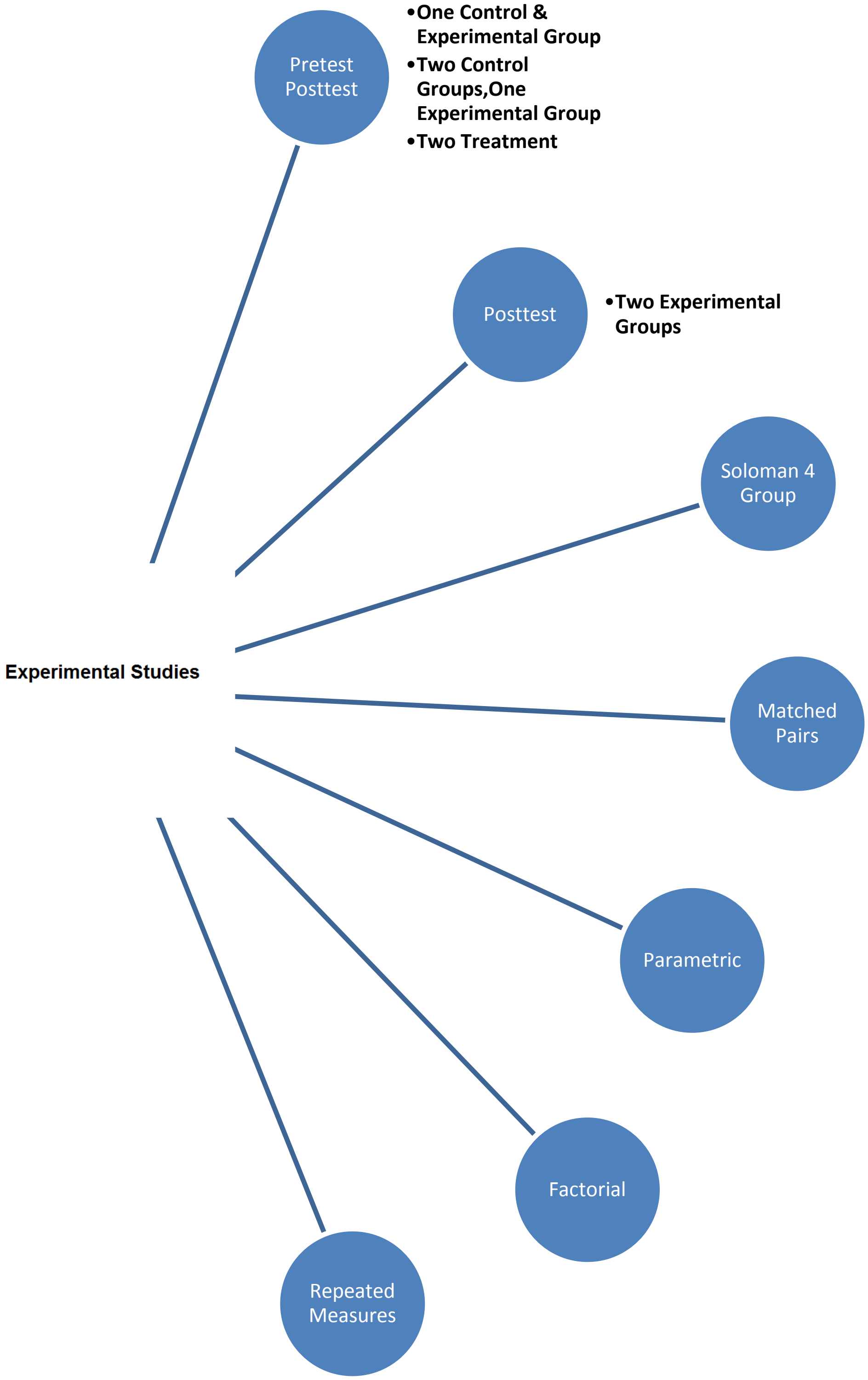


Figure 3.2 Types of Experimental Studies

3.2.1. A Brief History of Experimentation

Experimental research designs have been very prominent in the fields of natural science with renowned researchers like Galileo implementing experimentation methodology in their quest for answers. However, in the social sciences, experimental designs have been used rarely and infrequently (Bositis & Steinel, 1987).

3.2.1.1. Historical Overview & 21st Century Trends in Experimental Research in Education

One of the biggest scientific developments of the 20th century was the beginning of use of experimental research methods applications to questions about human behavior and cognition (Odle & Mayer, 2006).

The three decades 1975-2005 saw a mega bloom in experimental research in education and contributed significantly on instruction design and other areas for reading, writing, science and mathematics (Odle & Mayer, 2006).

Despite these developments a research about trends in educational research (Hsieh, et al., 2005) found a decline in the percentage of articles based on experimental studies (dropped to 26% in 2004 as compared to 40% in 1983 in primary educational psychology journals). This downward trend is attributed to a lack of training in experimental research methods in schools of education. This sidelines educational researchers by depriving them of one of the most potent and prolific research methodologies (Odle & Mayer, 2006).

Due to concerns about the quality of research in education, a law was passed by the U.S. Congress, necessitating educational research studies in the United States to be based on "scientifically-based research" (Reyna, 2005, p. 30) . Scientifically based research includes "experimental or quasi-experimental designs…with appropriate controls to evaluate the effects of the condition of interest" (Reyna, 2005, p. 38).

(Shavelson & Towne, 2002, p. 3) appealed to the National Research Council calling for "evidence-based research" in education. This is described as the testing of hypotheses against empirical evidence as opposed to testing it against a theory, an ideology or random observation.

The start of the 21st-century saw the use of effect size, meta-analysis, net impact and randomized field trials (RFT) in educational research (Odle & Mayer, 2006). The most noticeable was the increasing trend of effect size, with a significant increase in the studies using it (from 4% in 1995 to 61% in 2004). From 1995 to 2004 the rate was stable at 25% (Hsieh, et al., 2005).

To conclude, the experimental research methodology that led to a boom of scientific research about humans in the 20th century can serve as a potent and powerful tool in educational researches in the 21st century (Odle & Mayer, 2006).

3.2.2. Procedures for Experimental Research

Experts have listed an ideal route, a set of procedures to be followed while conducting an experimental research study. However, educational research tends to proceed in an irregular way with many unforeseen circumstances arising at various stages of the study. Therefore, these measures only serve as guidelines and cannot be followed religiously. They are listed below (Cohen, Manion, & Morrison, 2007):

1. **Identify and define the research problem:**

    This is the first step of the study and determines the direction and scope of the experiment. Therefore, it must be clear and specific.

2. **Select relevant variables.**

    Variables are characteristics that are not fixed (vary) and must be measurable. In certain situations, a variable is not directly measurable. It needs to be defined by substituting another characteristic (that is measurable) to stand in place of the first characteristic. The measurable variable (known as the proxy variable) must be a valid indicator of the unmeasurable variable (known as the hypothetical variable). Thus, the

hypothetical variable becomes operationally defined and can be used in the experimental study.

This stage involves two steps:

- Predicting the relationships between variables using logic, previous researches, theory or observation.
- Prioritizing between variables; deciding which ones are crucial for the research problem and need to be varied experimentally, whereas which ones are not relevant and need to be excluded by means of controls.

3. **Formulate hypothesis:**

A hypothesis precisely states the specific relationship expected to exist between the variables (ETSU, 2015). It is based on the findings of the previous stage.

4. **Choose suitable experiment (type of study):**

The most appropriate type of study needs to be chosen depending on the research problem, issues of practicality and availability of resources.

5. **Select levels at which to test independent variables:**

Each variable has its unique characteristics and the level at which it is to be measured to make a meaningful observation.

6. **Pilot test:**

The pilot test is an initial small scale trial of the experiment to identify possible problems or issues in the experiment.

7. **Conduct tests (pre-test, post-test) when and if needed.**
8. **Implement the intervention.**

In the implementation, care should be taken to meticulously follow the previously tested instructions, timing and sequence.

9. **Choose suitable methods of analysis.**
10. **Analyze data and conclude findings.**

Figure 3.3 summarizes the procedures for experimental research.

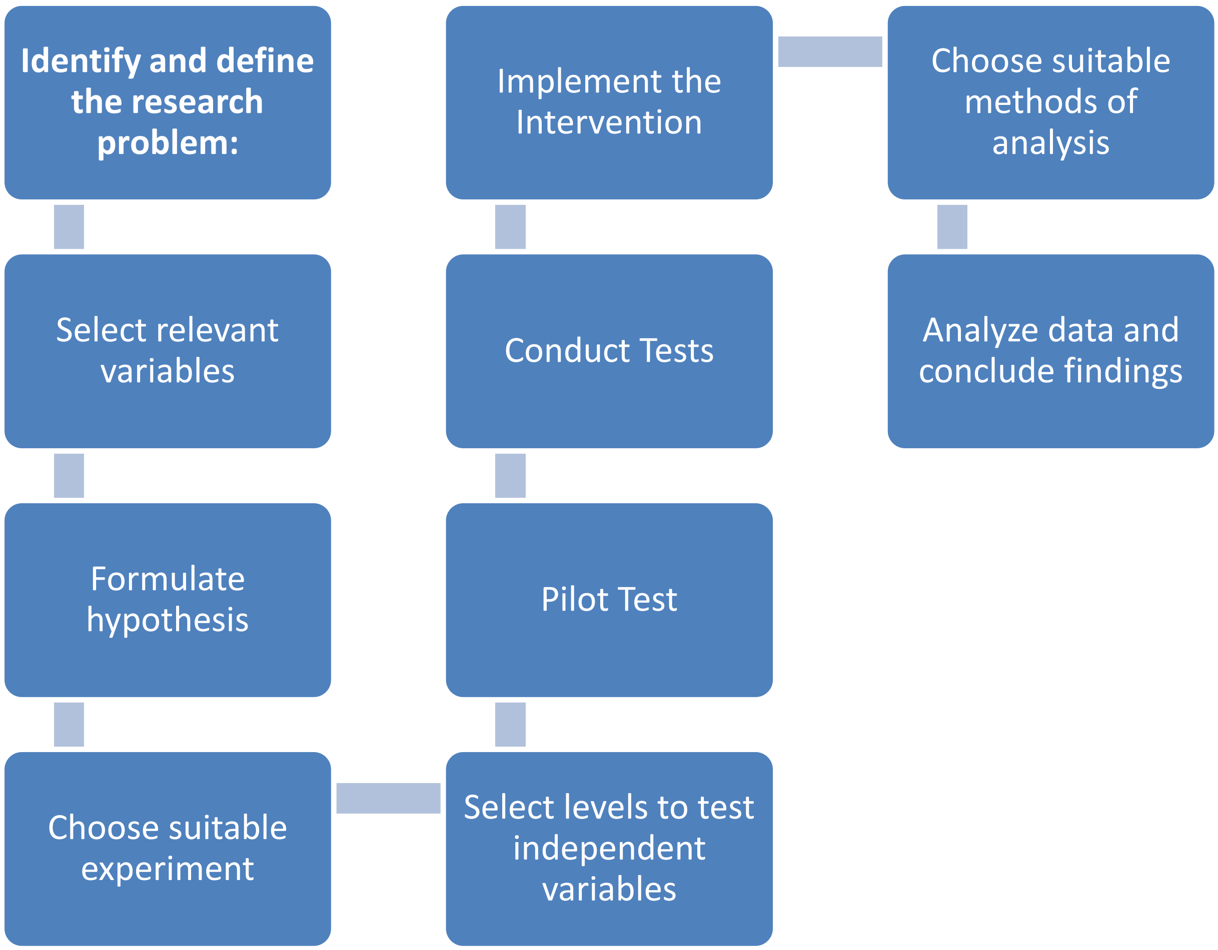


Figure 3.3 Procedures for Experimental Research

3.2.3. Experimental Research-Benefits and Limitations

Numerous researchers claim that to draw conclusions about instructional interventions and to determine if a particular strategy results in a subsequent increase in learning, experimental research designs are the most suitable method (Odle & Mayer, 2006).

The findings of the Committee on Scientific Principles for Education Research (2002) set up by the National Research Council thoroughly analyzed education research methods and concluded that randomized trials and experimental research are the most suitable method for comparison between variables, therefore ideal for determining the change in an outcome (Shavelson & Towne, 2002).

Other researchers have labelled experimental methods, involving treatments (intervention) while keeping other variables controlled as the “gold standard” for educational psychology since the evolution of the field (Phye & D H Robinson, 2005). Whereas (Phye & D H Robinson, 2005) believe that the research world is at the edge of a confirmation that experimental research strategies provide the most credible and sound evidence.

However, experimental research has some limitations. These limitations are two-fold: theoretical and methodology related.

In theory, experimental research enables us to make a judgement about the superiority of one method of instruction over conventional practice, but it may be unable to specify the exact mechanisms that caused the enhancement in student learning.

While designing the methodology, a controlled environment (that allows variation in one factor while all others are kept constant) is created. While this is a necessary condition for the experiment to be conducted and the measurement of meaningful results, such an environment is not a replica of a regular educational setting, nor is it possible for these conditions to be imposed in the real world. Thus a compromise has to be made between experimental precision and practical reality (Odle & Mayer, 2006).

3.3. Quasi-experimental Study

Quasi means: “partly; almost” (Press, 2015). Therefore a quasi-experimental study is a study that has most, but not all the features of an experimental study. It is a close variant of an experimental research design. Quasi-experimental or field research is conducted in a natural setting as opposed to a laboratory set up. The variables are isolated, controlled and manipulated to suit the needs of the research question. (Cohen, Manion, & Morrison, 2007). However, the researcher does not exercise complete control over the research setting (experimental conditions or extraneous variables) as in an experimental design (Wolf, 2005). (Kerlinger, 1979) describes quasi-experimental studies as ‘compromise designs’.

3.3.1. Types of Quasi-experimental Research Designs

Quasi-experimental research designs can take the following forms (Cohen, Manion, & Morrison, 2007):

- Pre-experimental designs:
    - the one group, pre-test and post-test design;
    - the one group, post-test only design;
    - the post-tests only, nonequivalent design.
- Pre-test and post-test, nonequivalent, control and experimental group design.
- One-group time series.

Figure 3.4 summarizes the types of quasi-experimental research designs.

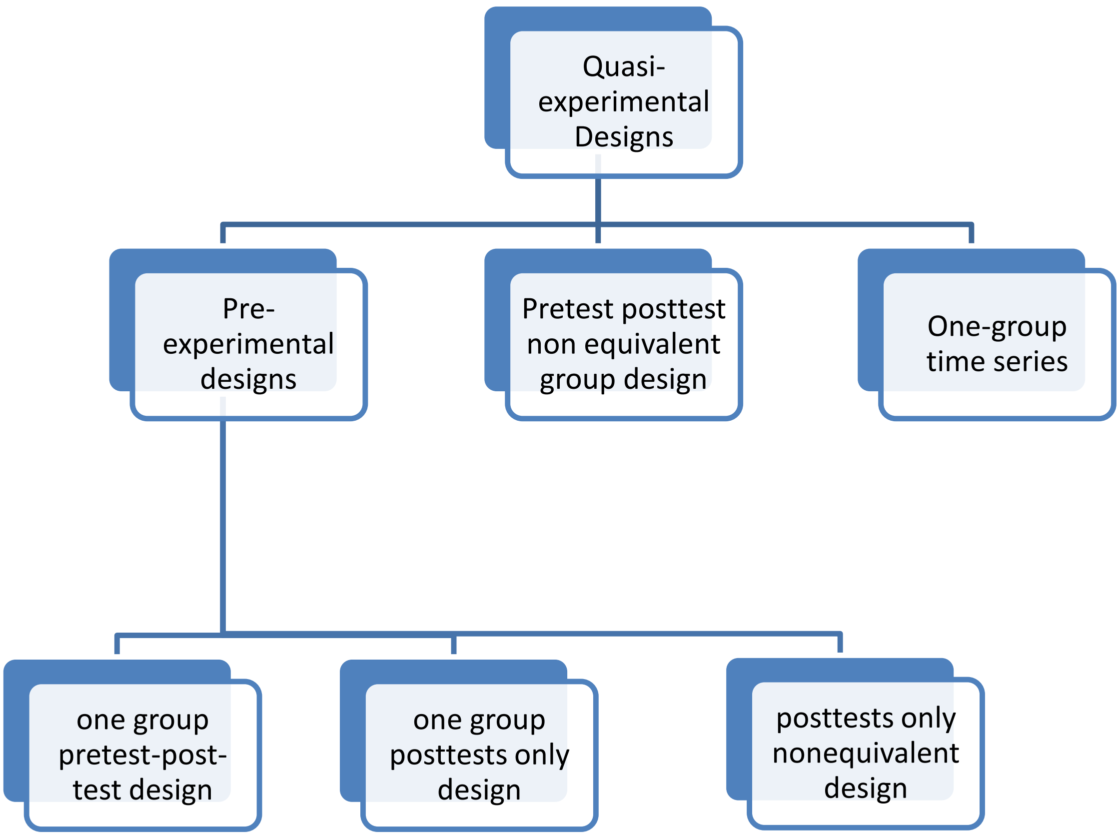


Figure 3.4 Types of Quasi-experimental Research Designs

3.3.2. The Pre-test Post-test Control and Experimental Group Design

For the purpose of this project, focus will be on the Pre-test Post-test Control and Experimental group design.

3.3.2.1. Representation

The pre-test post-test control group design can be represented in the following way (Cohen, Manion, & Morrison, 2007):

Experimental	**NR**	**O_1**	**X**	**O_2**
Control	**NR**	**O_3**		**O_4**

Where, NR denotes non-randomization;

O_1 denotes pre-test;

X denotes the treatment implemented (independent variable);

O_2 represents post-tests.

Both the control and experimental group undertake the pre-test and post-test, only the experimental group receives the research treatment.

3.3.2.2. Procedure

Experts have listed an ideal route, a set of procedures to be followed while conducting an experimental research study. However, educational research tends to proceed in an irregular way with many unforeseen circumstances arising at various stages of the study. Therefore, these measures only serve as guidelines and cannot be followed religiously. They are listed below (Cohen, Manion, & Morrison, 2007):

1. **Identify and define the research problem.**
2. **Select relevant variables.**
3. **Formulate hypothesis.**
4. **Choose suitable experiment (type of study).**
5. **Select levels at which to test independent variables.**
6. **Conduct pre-test.**
7. **Implement the intervention.**
8. **Conduct post-test.**
9. **Choose suitable methods of analysis.**

Figure 3.5 enlists the procedures for conducting a quasi-experimental research study.

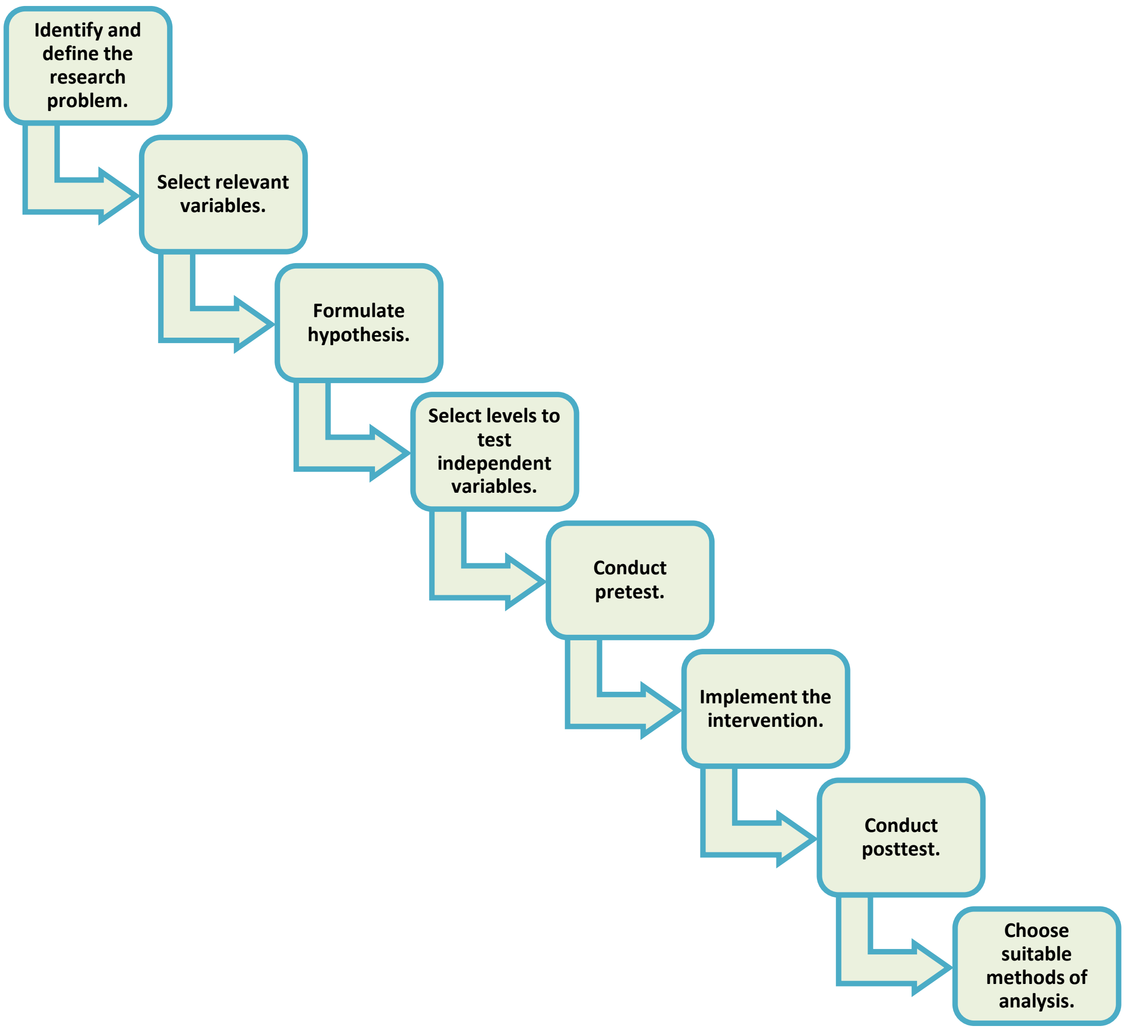


Figure 3.5 Procedures for Conducting a Quasi-experimental Research Study

3.3.2.3. Association and Causation

As defined earlier, research is the study of relationship among different variables. The relationship between two variables can be of three types (Wolf, 2005):

- **Association**: is a connection between variables, where one varies according to variations in the other (Payne & Payne, 2004). The first variable may or may not be the cause of changes in the second one.
- **Causation**: is a connection between variables, where the change in one variable is the reason for change in another variable.
- **Both** association and causation.

Experimental studies can substantially establish the presence of causal relationships between variables. In quasi-experiments, the effect is measured after the cause has taken place, therefore the cause can be influenced and manipulated (Shadish, Cook, & Campbell, 2002)**.**

3.3.3. Benefits and Limitations of Quasi-experimental Designs

Less Resources and No Ethical Concerns:

Since quasi-experimental designs do not use the process of randomization for group allotment of participants, they are easier to set up and require less resources and time. Moreover, there are no ethical concerns of deliberate withholding of treatment for controlled group, or purposely endangering the safety or health of either group.

Minimal Threat to External Validity:

External validity is the degree to which the findings of a research can be generalized to the wider population, setting, measurement and treatment variable (Cohen, Manion, & Morrison, 2007). In a quasi-experimental design, threats to external validity are minimized as

the research setting is less artificial as compared to a laboratory experiment. The results of a quasi-experimental study are therefore more generalizable.

Lack of Control on Extraneous Variables

Lack of randomization process increases the chances of extraneous variables influencing the results of the study.

A meticulously designed quasi-experiment can maximize the advantages and minimize the limitations, thus, resulting in a strong research study with valid and reliable findings.

4. Methodology

4.1. Overview

The basic idea of this study was to use visualizations to check if it deepened the conceptual understanding and accentuated the adaptive reasoning of Calculus for high school students. The focus was on the topics of Integration and Differentiation. A Quasi-experimental study was conducted. The pre-test post-test control and experimental group design was employed.

4.2. Group Allocation

Since the purpose of this project was to compare if the use of visualization had a significant effect on the conceptual understanding and adaptive reasoning of Calculus, the quasi experiment was conducted on students who had studied mathematics (calculus) at the high school level and already completed high school.

The students of School of Electrical Engineering and Computer Sciences at National University of Sciences and Technology were chosen for the control and experimental groups.

In order to set up an environment for a fair comparison, two sections from the second semester of Bachelors of Science in Computer Sciences were selected and were assigned as the control and experimental group. Choosing from same discipline/field ensured that their exposure to university level calculus was uniform. To further ensure uniformity, both sections selected were being taught calculus by the same professor.

Both groups had a mixture of male and female students. Overall the males were in majority, but the ratio of male: female participants in each group was the same.

4.3. Group Size

The Control and Experimental groups were of size 33 each. Table 4.1 summarizes the demographics of the participants.

Table 4.1 Demographics of Participants

	Control Group	**Experimental Group**
Current education level	BS (CS), 2nd semester	BS (CS), 2nd semester
Total No of participants	33	33
Gender Distribution	7 females, 26 males	6 females, 27 males

4.4. Procedure

4.4.1. Identifying and defining the research problem

This study was an attempt to identify and present pedagogical techniques, supported with technology that can substantially increase two of the five strands of mathematical proficiency i.e. Conceptual Understanding and Adaptive Reasoning. Given the benefits of using illustrations, diagrams and visual aids in education established by previous researches, this project was designed to investigate if the use of visualization has a significant effect in calculus teaching. Thus the following research question was formulated:

Research Question: How does visualization impact the conceptual understanding and adaptive reasoning of high school calculus students?

4.4.2. Selecting relevant variables

The independent and dependent variables were identified as follows:

Independent Variable: Visualization

Dependent Variables: Conceptual Understanding and Adaptive Reasoning

4.4.3. Hypothesis formulation

The following Null Hypothesis was formulated:

Hyp $_0$: Use of visualization as a supplement to teaching has no significant effect on the *Conceptual Understanding* and *Adaptive Reasoning* of calculus for high school students.

The Alternative Hypothesis was defined as follows:

Hyp $_A$: Use of visualization as a supplement to teaching has a significant effect on the *Conceptual Understanding* and *Adaptive Reasoning* of calculus for high school students.

4.4.4. Choosing suitable experiment (type of study)

Pre-test Post-test Control and Experimental Group Design was chosen. The nonequivalent pre-test post-test control group design is labeled as "one of the most commonly used quasi-experimental designs in educational research" (Cohen, Manion, & Morrison, 2007, p. 283). This is the most suitable design for the decided upon research question, for the reasons listed below:

4.4.4.1 Conducting a pre-test:

A pre-test evaluates the level of the participants at the start of the quasi experiment and ensures that there is actually a need for this intervention. In the case of this study, the pre-test

results showed that there in fact was a lack of conceptual understanding and adaptive reasoning among the participants.

Had the results been different, the pre-test would have determined that there was no need of an intervention to increase conceptual understanding and adaptive reasoning; or that this particular group of participants was not suitable for this intervention.

4.4.4.2. Control and Experimental Groups:

The purpose of a control group is to determine what outcome will occur in case the intervention (independent variable) had not taken place. This control group should be as similar to the experimental group as possible, in as many dimensions as possible. The presence of the control group ensures that the effect of extraneous variables are controlled for and this makes the experiment strong.

4.4.4.3. Post-test:

The post-test results of both control and experimental groups show the changes (if any) from the intervention.

4.4.5. Selecting levels to test dependent variables

The independent variables (Conceptual Understanding, Adaptive Reasoning) were tested by using pre-test and post-tests. A quiz of 10 questions was designed, with a maximum score of 15 and the total time allowed was 10 minutes.

4.4.6. Designing and Implementing the Pre test

As mentioned in the previous heading, the pre-test comprised of 10 main questions and sub questions. It was designed to test the Conceptual Understanding and Adaptive Reasoning of the participants. The questions were designed in English, the official medium of instruction in the institute that the participants belonged to. The language used was simple and the

questions posed consisted of scenarios from daily life, with little to no emphasis on procedural drill (mathematical calculations).

The questions were structured in the following way:

- A practical scenario was described- the relevant circumstances listed.
- Decision: The participant had to make a decision (choose from a list of options, or decide which mathematical operation should be applied in the given circumstances). This step aimed at evaluating the conceptual understanding.
- Justification and Reasoning: The participant had to explain *why* they made the choice in the previous step.

To test Conceptual Understanding, the questions were based on real life scenarios, and tested whether or not the participant understood what mathematical symbols and procedures mean.

To test Adaptive Reasoning, the questions designed required the participants to explain and justify their answer. This practice builds the understanding of math because the student is consciously thinking about the reason for choosing a particular solution (Danley, 2008).

For the pre-test and its marking scheme, see Appendix A

4.4.7. Designing and Implementing the Lecture and Visualization

The implementation for both the Control and Experimental Group is listed in Table 4.2.

Table 4.2 Implementation

	Control Group	Experimental Group
Introduction	5 minutes	5 minutes
Pre-test	10 minutes	10 minutes
Lecture	20 minutes	20 minutes
Visualization	-	10 minutes
Post-test	10 minutes	10 minutes

Lecture:

The 20 minute lecture was delivered (to both the Control and Experimental groups) by Dr. Sohail Iqbal, Assistant Professor and Head of Department of Computing at School of Electrical Engineering and Computer Sciences. He has over 12 years of teaching experience and holds a doctorate in Surgical Robotics from University of Paris-Est, France.

The contents of the lecture were as follows:

- A brief history of Calculus
- Why study Calculus?
- What it means to “Integrate”
- What it means to “Differentiate”
- Practical Applications of Calculus

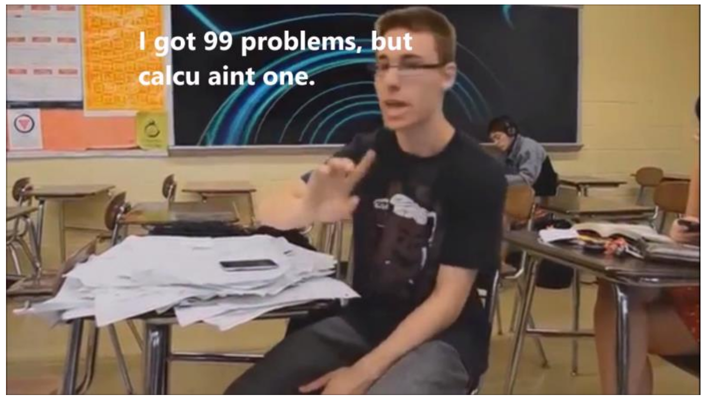
Visualization:

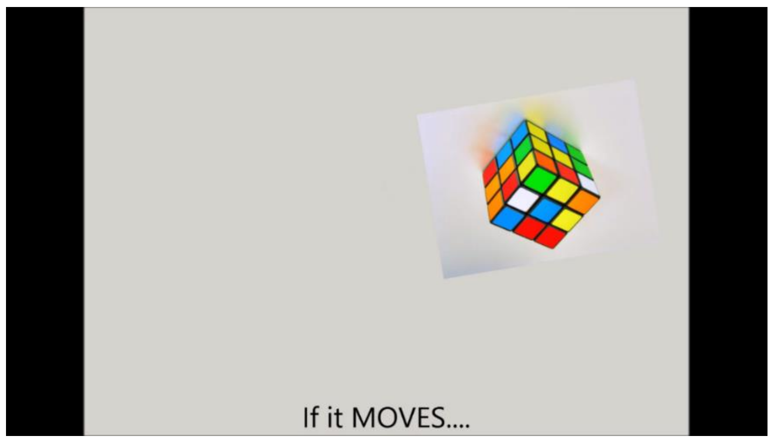
Based on the researches and recommendations of David Tall (Tall, 1990) (Tall, 1992) (Tall & Sheath, 1983) and Abraham Arcavi, (Arcavi, 2003, revised 2006) the design of the visualizations aimed at showing a broader view of and capturing the dynamic nature of mathematical concepts, relations and operations.

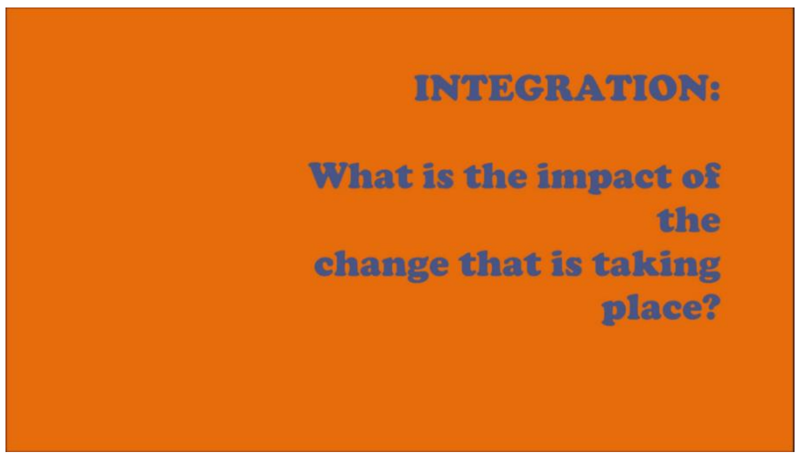
Link for viewing visualization: https://www.youtube.com/watch?v=XTLVm5aXdnI
http://www.dailymotion.com/video/x31p0sp

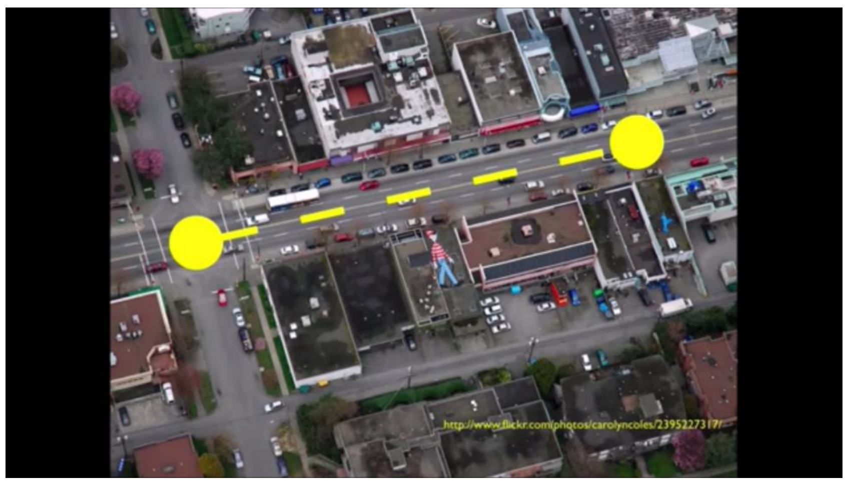
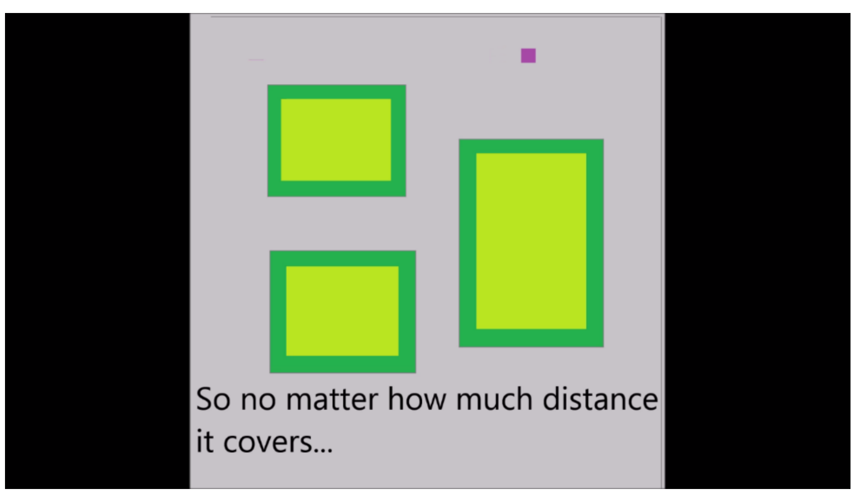
Storyline for Visualization (with Screenshots):

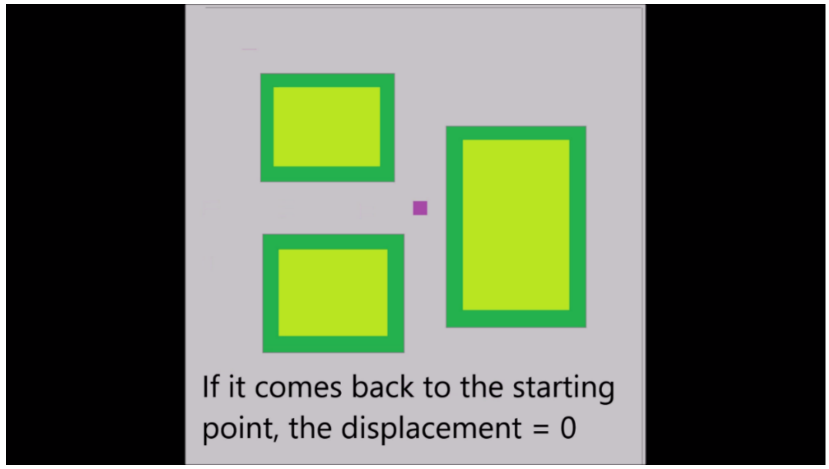
Figure 4.1 Storyline of Visualization

Rap Song by a boy who hates calculus	
Calculus-a tool to understand change.	

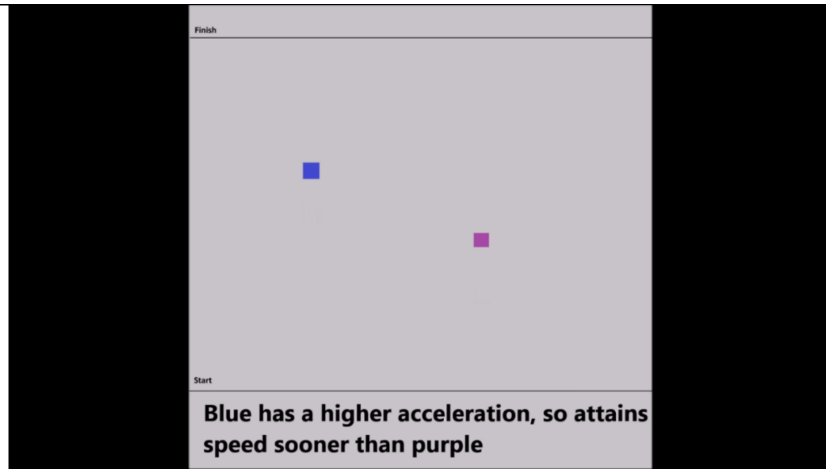
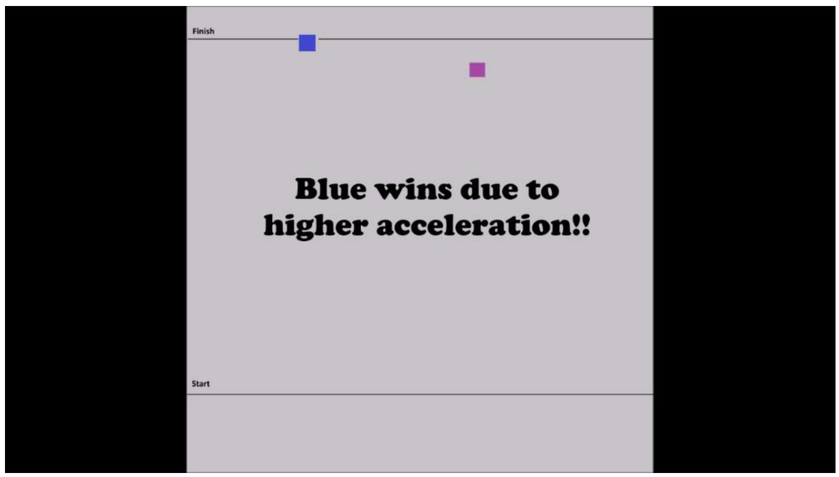
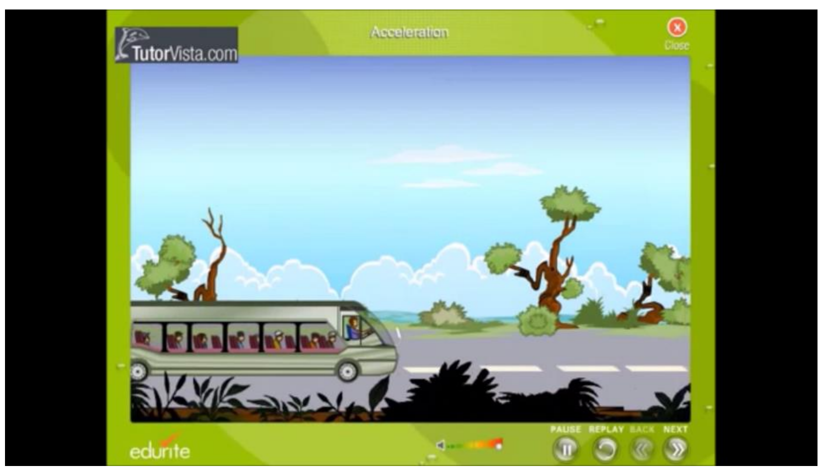
	
Integration and Differentiation-what they actually mean, in layman terms.	

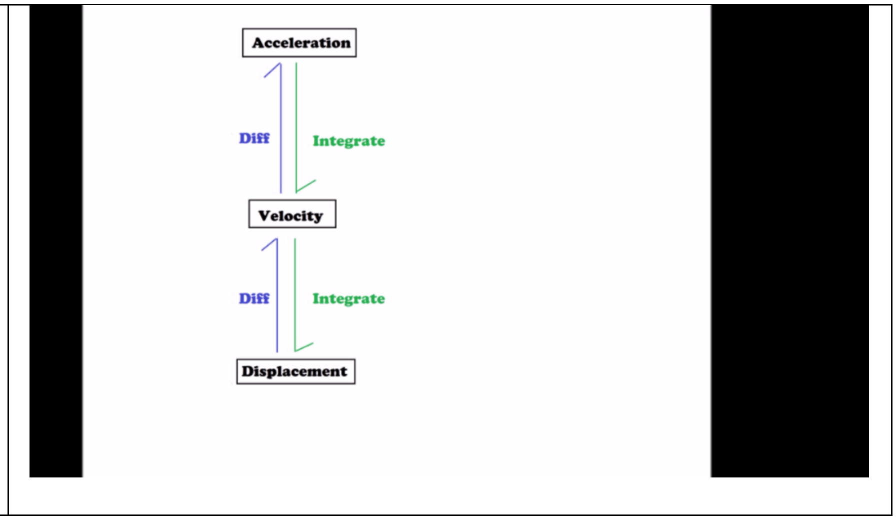
	 
Integration is Anti-Differentiation.	

	
Animations to explain position, displacement, velocity, acceleration and their relationship.	
Difference between distance and displacement explained by a moving object and its trajectory.	

	 If it comes back to the starting point, the displacement = 0  http://www.flickr.com/photos/epac_island/455837464/  http://www.flickr.com/photos/hamed/282241413/

	http://www.flickr.com/photos/dawvon/32305882/ 80 60 100 40 120 20 140 mph km/h 160 53201 348.5 http://www.flickr.com/photos/philon/515535717/in/set-72157600270761318/

	Finish Start Blue has a higher acceleration, so attains speed sooner than purple Finish Blue wins due to higher acceleration!! Start TutorVista.com Acceleration Close edurite PAUSE REPLAY BACK NEXT



4.4.8. Conducting post-test

The post-test was designed to be very similar to the pre-test. The only difference was that the sequence of the questions was rearranged a bit and the names and numbers in the question statement were changed. The level of difficulty, the nature of the question and distribution of questions was exactly the same as the pre-test. For the post-test and its marking scheme, see Appendix B

4.4.9. Choosing suitable methods of analysis.

Statistical analysis requires choosing the appropriate statistical test. Figures 4.2 and 4.3 show how this is done.

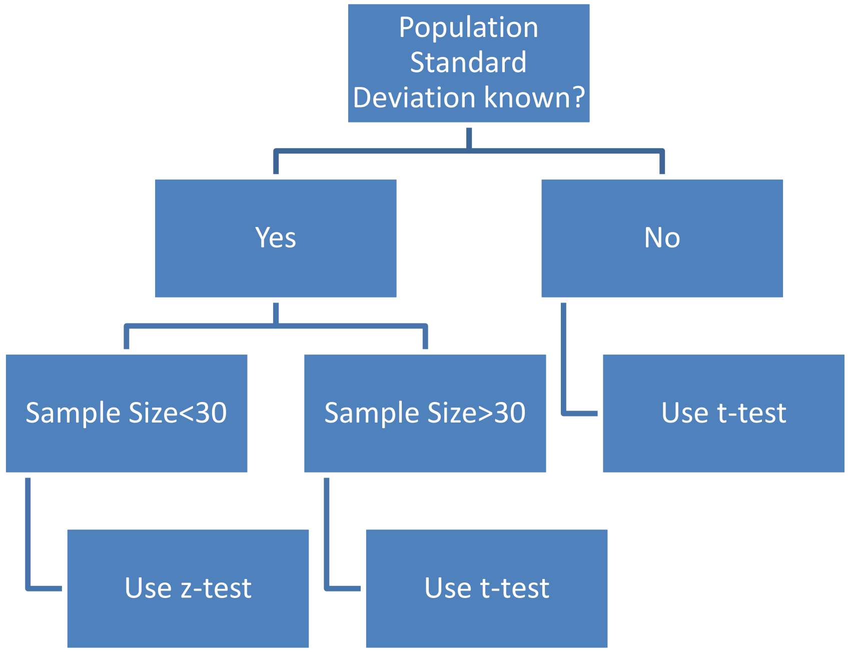


Figure 4.2 T-Test vs Z-Test

Image adapted from (Statistics How To, 2015)

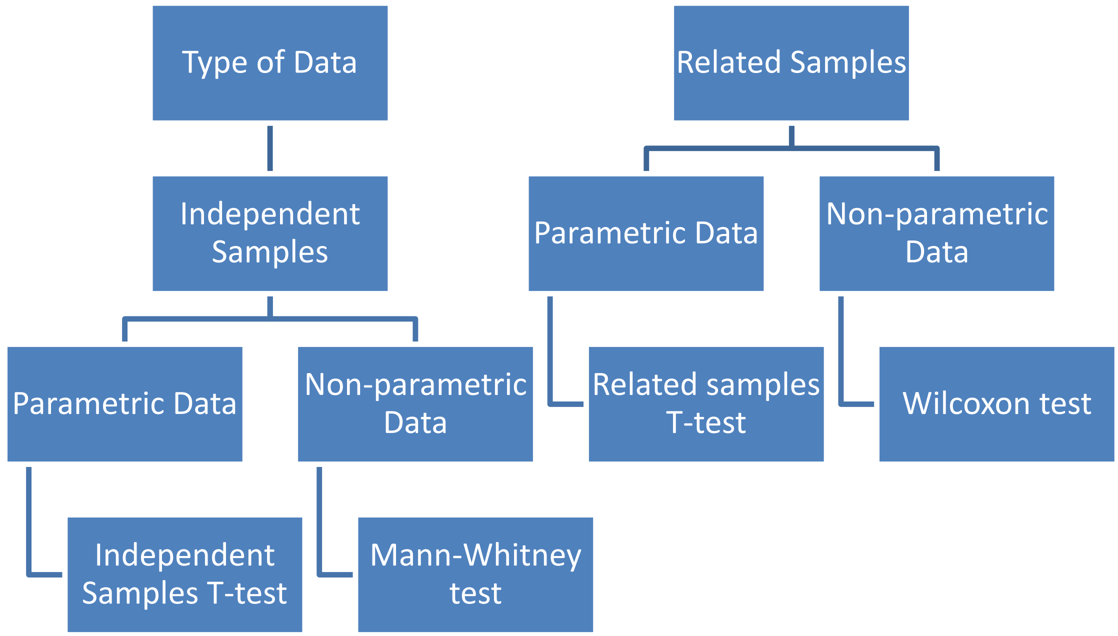


Figure 4.3 Which T-test to Use

Image adapted from (Greasley, 2007)

4.5. Summary

A pre-test post-test control and experimental group design was employed in the study. A group size of 33 each was used. The control and experimental groups were two sections of BS CS 4 in NUST-SEECS, who were taught Calculus by the same instructor.

The research question explored was "*How does visualization impact the conceptual understanding and adaptive reasoning of high school calculus students*?". The independent variable was Visualization and the dependent variables were Conceptual Understanding and Adaptive Reasoning. The null hypothesis was "*Use of visualization as a supplement to teaching has no significant effect on the Conceptual Understanding and Adaptive*

Reasoning of calculus for high school students” whereas the Alternative Hypothesis was *“Use of visualization as a supplement to teaching has a significant effect on the Conceptual Understanding and Adaptive Reasoning of calculus for high school students”.*

The pre-test was conducted for both groups followed by a 20 minute verbal lecture by an experienced Calculus instructor. After the lecture, the visualization was shown to the experimental group only. Lastly, a post-test was conducted for both groups.

An independent samples T-test was used to investigate if the difference between the mean scores of the post-test were significantly different for both groups or not; and the effect size was calculated. The results and findings of the study are explained in detail in the next chapter.

5. Data Analysis and Results

5.1. Overview

Relevant data was collected before, during and after the implementation phase of the design experiment. The collected data was then tabulated and statistical tests were conducted, the results of which were used in analyzing the data.

5.2. Data Collected

5.2.1. Pre-test Data

5.2.1.1. Control Group Pre-test Data

The relevant aspects of the Control Group Pre-test Data are listed in Table 5.1.

Table 5.1 Control Group Pre-test Data

Mean	6.44
Mean Percentage Score	(6.44/15)*100 = 42.9%

5.2.1.2. Experimental Group Pre-test Data

The relevant aspects of the Experimental Group Pre-test Data are listed in Table 5.2.

Table 5.2 Experimental Group Pre-test Data

Mean	6.62
Mean Percentage Score	(6.62/15)*100 = 44%

5.2.2. Post-test Data

5.2.2.1. Control Group Post-test Data

The relevant aspects of the Control Group Post-test Data are listed in Table 5.3.

Table 5.3 Control Group Post-test Data

	Statistic	**Standard Error**
Mean	7.91	0.262
95% Confidence Interval (Lower Bound)	7.37	
95% Confidence Interval (Upper Bound)	8.44	
5% Trimmed Mean	7.92	
Median	8.00	
Variance	2.273	
Standard Deviation	1.508	
Minimum	5	
Maximum	11	
Range	6	
Interquartile Range	2	
Skewness	-0.069	0.409
Kurtosis	-0.538	0.798

5.2.2.2. Experimental Group Post-test Data

The relevant aspects of the Experimental Group Post-test Data are listed in Table 5.4.

Table 5.4 Experimental Group Post-test Data

	Statistic	Standard Error
Mean	10.30	0.273
95% Confidence Interval (Lower Bound)	9.75	
95% Confidence Interval (Upper Bound)	10.86	
5% Trimmed Mean	10.28	
Median	10	
Variance	2.468	
Standard Deviation	1.571	
Minimum	7	
Maximum	14	
Range	7	
Interquartile Range	2	
Skewness	0.389	0.409
Kurtosis	0.159	0.798

5.3. Effect Size

Effect size is the extent to which the phenomenon being investigated is present, or the degree to which a null hypothesis is not supported. It tells how big the effect of the intervention is (Wright, 2003)

Effect size is an easy yet powerful way of measuring the difference between two sets of data, as it is sensitive to even those values that fall below the standard cut off points (Cohen, Manion, & Morrison, 2007).

Effect Size can be measured in the following way:

(difference between means of experimental group and control group) / standard deviation of the control group

Effect Size ranges indicate:

0–0.20 = weak effect

0.21–0.50 = modest effect

0.51–1.00 = moderate effect

>1.00 = strong effect

5.3.1. Effect Size Calculations

(10.31 - 7.91)/ 1.508 = 1.59

Therefore, the effect size is 1.59. This is a **strong effect**.

5.4. T-test

5.4.1. Definition

A t-test is conducted to investigate if a difference between the means of two groups occurred because of coincidence or has a significant reason that is causing it.

A difference has a higher chance of being meaningful if:

1. There is a large difference between the means of both groups;

2. bigger sample size;
3. the standard deviation is low.

5.4.2. Related vs Independent

In educational research, the following terms are of relevance:

- **Related samples:** when two sets of data are obtained from the same people
- **Independent samples:** when two sets of data are obtained from different people.

Since this project comprised of a control and an experimental group, both containing different sets of participants and the two sets of scores (post-test scores from both groups) were provided by different people, thus it was an independent samples study.

5.4.3. Parametric vs Non parametric

Parametric tests are to be used when:

1. The distribution of the data obtained is similar to a normal distribution;
2. The variance of the data obtained of both groups similar (shows homogeneity of variance).

5.4.3.1. Tests for normality

The tests for normality compare the shape of the sample distribution of the data collected to the shape of a normal curve in order to check if the sample data is normally distributed or not.

To check for normality, the shape of the histogram can be observed for similarity with the normal distribution curve. The histograms for Control Post test scores and Experimental

Post test scores are shown in Figures 5.1 and 5.2 respectively.

ControlPosttest

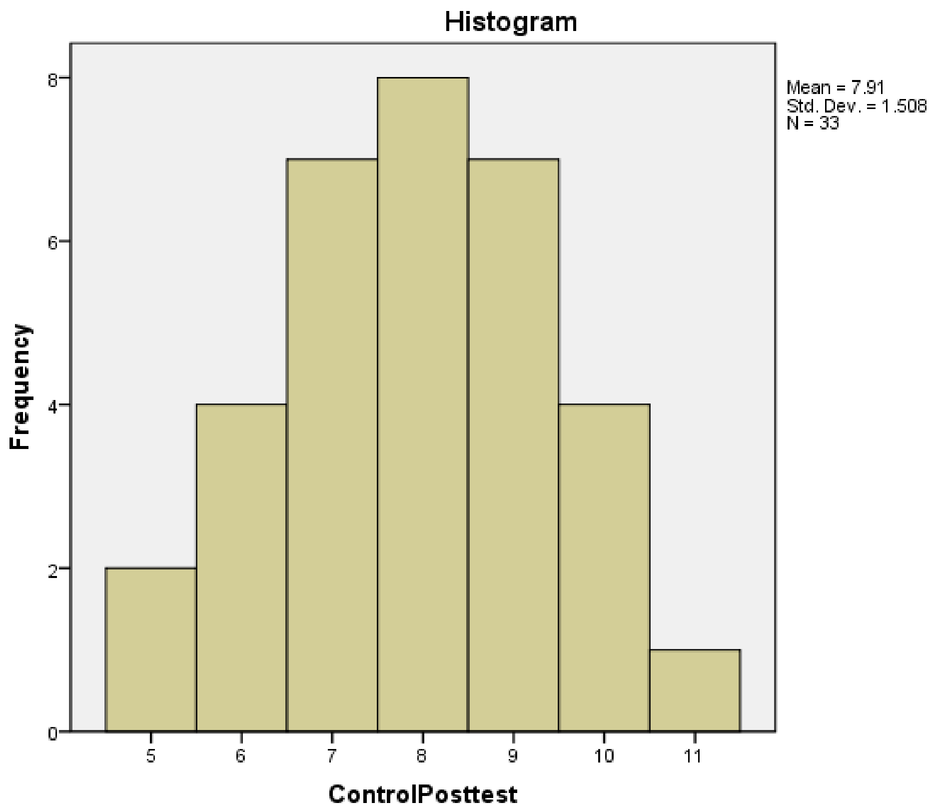


Figure 5.1 Histogram for Control Post test scores

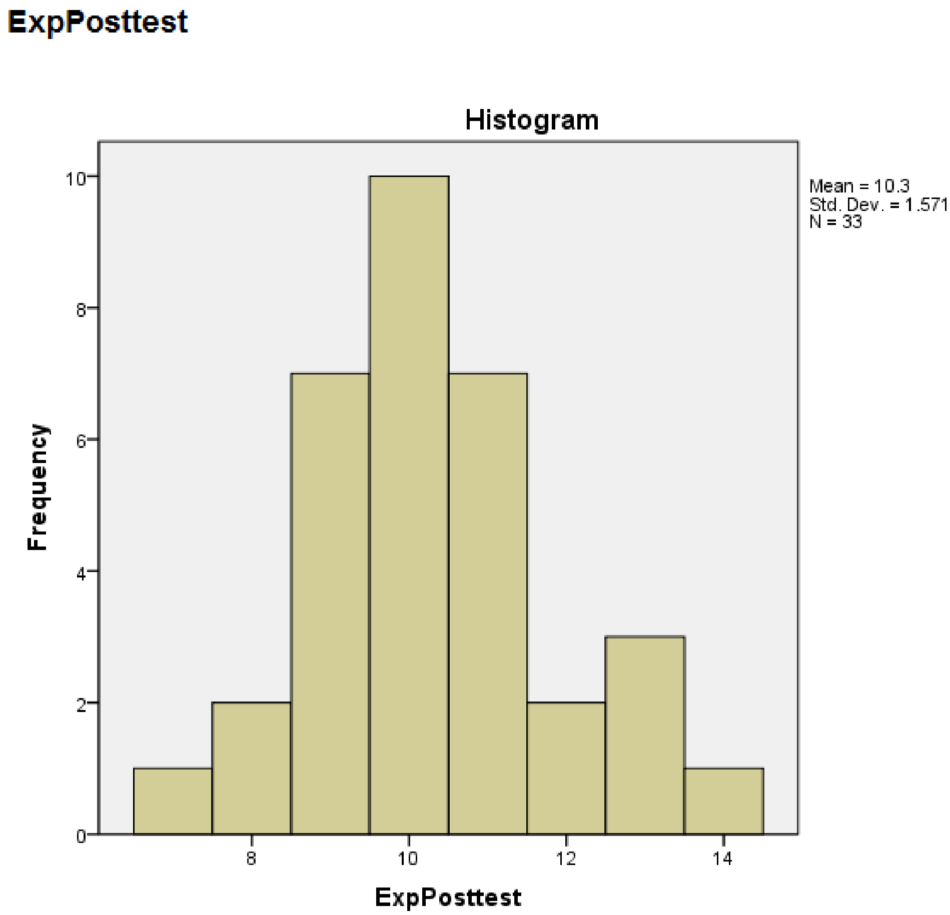


Figure 5.2 Histogram For Experimental Post Test Scores

However, this is not a very objective method. In SPSS, the **Shapiro Wilk** test can be used to objectively test for normality for sample sizes ranging from 0 to 2000. (For this project, the sample size is 33).

Shapiro Wilk tests the NULL hypothesis that the samples in fact come from a Normal distribution.

- W = 1 when sample-variable data are perfectly normal;
- For p-values (sig) <=0.05: the null hypothesis is rejected;
- For p-values (sig) >0.05: the null hypothesis is failed to reject.

Result of tests for Normality conducted in SPSS for Control Post test scores are listed in Table 5.5 below.

Table 5.5 Shapiro Wilk test for Control Post test scores

	Statistic (W)	df	P value (sig)
Control Post-test	0.958	33	.222

P-value (sig) for Control post test scores is > 0.05 and W is very close to 1. Therefore, it can be concluded that the null hypothesis is not rejected and the control post-test scores sample is normally distributed.

Result of tests for Normality conducted in SPSS for Experimental Post test scores are listed in Table 5.6 below.

Table 5.6 Shapiro Wilk test for Experimental Post Test Scores

	Statistic (W)	df	P value (sig)
Experimental Post-test	0.949	33	0.124

P-value (sig) for Control post test scores is > 0.05 and W is very close to 1.Therefore, it can be concluded that the null hypothesis is not rejected and the experimental post-test scores sample is normally distributed.

5.4.3.2. Homogeneity of variance

Homogeneity of variance means that the spread of score, or variance of two sets of data is approximately similar.

To test for homogeneity of variance: Levene’s Test was performed.

- If the Levene's test results in a p <= 0.05, the test is statistically significant and the variance of both sets of data is not similar (homogeneity of variance is not shown);

- If the Levene's test results in a p > .05, the test is statistically insignificant and the variance of both sets of data is similar (homogeneity of variance is shown).

Result of tests for Homogeneity of Variance conducted in SPSS for Control Post test scores and Experimental Post test scores are shown in Table 5.7 below.

Table 5.7 Levene's Test Results for Post Test Scores

Post test scores	0.004	0.949

The p-value for the Levene’s test is > 0.05, therefore it can be concluded that homogeneity of variance exists between both the Control Post test scores and Experimental Post test scores.

5.4.4. Choosing a T-Test

As the data comes from independent samples and is parametric (normality and homogeneity of variance have been established), therefore the independent samples T test was the most appropriate test for this project. (Refer to Figure 4.3).

5.4.5. Output of the T-test

The output of the T-test is statistical significance. **Statistical significance** is an indication of the variation or difference between sample averages, whether or not it is likely to represent an actual difference between populations.

The outputs of the T-test are listed in Table 5.8.

Table 5.8 Outputs of Independent Samples T-test

	t	df	Sig P value (2 tailed)	Mean Difference	Standard Error Difference	95% Confidence Interval of the Difference	
						Lower	Upper
Post-test Scores	6.316	64	0.000	2.394	0.379	1.637	3.151

Independent Samples T-test assumes the hypothesis that the difference in the means of the two sets of data (in this case Control Post-test score and Experimental Post-test score) samples is statistically significant, p=0.05.

The column headed Sig (2-tailed) shows the statistical significance.

For values of p<=0.05, the hypothesis is accepted.

95% Confidence intervals provide a range of values, which means that if the sampling was to be repeated 100 times, 95 times the result (difference in means of both groups) would lie within these values.

5.5. Experimental Group Improvement in Scores

The mean scores for the experimental group increased from 6.62 in the pre-test to 10.30 in the post-test. This was an increase from 44.1% to 68.67% i.e. 24.57% increase.

Moreover, the mean difference between the control group post-test scores and experimental group post-test score is 2.394 and the 95% Confidence Interval of Difference states a minimum

increase of 1.637 and maximum increase of 3.151. Therefore, the use of a 10 minute visualization resulted in an increase of at least 10.91% and at most 21%.

5.6. Summary

The data from the pre-test and post-test was collected from both the control and experimental group. A strong effect size of 1.59 was calculated. An independent samples T-test showed a significant difference between the post-test scores of both groups. The use of the designed visualization in the experimental group resulted in a minimum increase of 10.91% and a maximum increase of 21% as compared to the control group.

After analyzing the data the Null hypothesis is rejected and alternate hypothesis is failed to reject i.e. ***Using visualization as a supplement to teaching considerably increases the Conceptual Understanding and Adaptive Reasoning of high school calculus.***

6. Discussion

6.1. Findings of the Study

The mean of the pre-test scores for both groups was very similar (6.44 and 6.62) which shows that at the beginning of the experiment, both groups were at the same level of mathematical proficiency. Therefore, suitable to be a part of a quasi-experimental study.

The maximum obtainable marks were 15, so the mean score of pre-tests for both groups was very low (42.9% and 44%). This is alarming as the participants are all students who have opted for highly technical degrees at the undergraduate level and are enrolled in the top ranked university of the country. As the tests were designed to evaluate the conceptual understanding and adaptive reasoning of calculus, the poor results show that there is a lack of understanding as to why a mathematical idea is important, which context is it to be used in and a logical approach to the relationships of different mathematical functions. In a nutshell, integrated and functional grasp of the content and the ability to justify conclusions was missing.

The subsequent increase in the mean scores of the post-tests was a positive sign. In the control group, between the pre-test and post-tests, a 20 minute lecture was given to ensure that the concepts are recalled and the usefulness of calculus in practical life is identified. During this lecture, the whiteboard was used by the instructor to write and draw when needed. However, the increase in the post-test score was still very low (from 42.9% to 52%). This indicates that despite being presented with the facts in a way that focused on the use of calculus in the practical world, the significance of mathematical functions and their relationships; the students were unable to make the connections. When presented with a real life scenario and dilemma, they were unable to identify which mathematical procedure should be applied in this case and why.

The experimental group showed a substantial increase based on statistical tests and there was a significant difference between the post-test score means of the control and experimental groups. This indicates the benefits of using visualization. The content matter covered in the verbal lecture and the visualization was essentially the same. The difference was in the way the facts were presented. In the verbal lecture, the information was delivered orally, with a slight reliance on visual cues in the form of static hand drawn diagrams on the board. In the visualization, the Rule of Four was applied, so the information was simultaneously being presented Verbally, Analytically, Numerically and Graphically. This simultaneous display of information through all four means of representation gave a holistic view of the concept being taught. Moreover, the animations and movements enabled the students to make intuitive and meaningful connections of concepts. Thus, deepening their conceptual understanding and increasing their adaptive reasoning.

6.2. Limitations

The results of this quasi-experimental research study support the use of visualization in mathematics education. However there are a few limitations of this study.

Firstly, due to time constraints, only the topic “Integration and Differentiation” was taught to both control and experimental groups (the former through a verbal lecture only and the latter through the same verbal lecture followed by a visualization). A more comprehensive study conducted in greater detail should include more topics so that a stronger conclusion about the benefits of visualization can be made.

Secondly, as the intervention was conducted during one lecture session (50 minutes), the implementation phase was very short. The claim of visualizations advantages could be strengthened if the intervention was conducted for a longer period of time.

Thirdly, both the control and experimental groups comprised of participants who were enrolled in a technical undergraduate degree in the top ranked university of the country. They

had secured admission after fulfilling the criteria of a good high school score/grades and a competitive entrance examination. So it is safe to assume that their mathematical skills were polished to say the least. A similar study should be conducted with participants with average and low scores/grades at the high school level.

Fourthly, the focus of this study was on two of the five strands of mathematical proficiency i.e. Conceptual Understanding and Adaptive Reasoning. For a holistic investigation, the other three strands i.e. Procedural Fluency, Strategic Competence, and Productive Disposition must be considered as well.

7. Conclusion

Previous researchers have stated the need for reforms in mathematics education and calculus has been identified as a particularly demanding field as it is considered esoteric and obscure. This study was undertaken due to a lack of research conducted in the field of technology usage in education, to identify the effectiveness of various technologies for the enhancement and betterment of education in general and mathematics education in particular. The theme chosen was a relatively less employed and even less researched mode of information representation i.e. visualization. This theme was based on the significance of vision as the most important sensory tool for input. Keeping in mind the limitations and inadequacy of static pictures, illustrations and diagrams pointed out by previous researchers, and to encompass the dynamic nature of the relationships of mathematical concepts, video and animations were used.

The effect on mathematical learning of using visualization in the classroom, as an add-on to the lecture, was observed. The five strand model presented by (J Kilpatrick, 2001) was used as the basis for "Mathematical Proficiency". Two of the five strands, conceptual understanding and adaptive reasoning were focused on.

Due to the complex and abstract nature of certain mathematical topics, the use of visualizations and animation can significantly increase the levels of learning as compared to conventional teaching practices. Using visualization as a supplement to teaching helps the student form a clearer mental picture of the topics taught, therefore increases their mathematical proficiency.

The null hypothesis was "Use of visualization as a supplement to teaching has no significant effect on the *Conceptual Understanding* and *Adaptive Reasoning* of calculus for high school students. Whereas, the Alternative Hypothesis was "Use of visualization as a

supplement to teaching has a significant effect on the *Conceptual Understanding* and *Adaptive Reasoning* of calculus for high school students".

A quasi (quasi meaning 'partly or almost') experimental study was conducted to investigate the potential effects of visualization when used as a teaching tool in the classroom. The pre-test post-test control and experimental group design was employed. The group size was 33 each. A comparison was drawn up between a group that was taught through a lecture (control group) and another that was taught through the same lecture but were shown a visualization that represented the contents of the lecture through video. In order to set up an environment for fair comparison, the two groups were chosen from the same field and level of education. Furthermore, the pre-test score was used to determine that both groups were at the same level of mathematical proficiency at the start of the experiment, therefore suitable to be used in a quasi-experimental study

Quantitative analysis was conducted on the data gathered from the research study and a significant difference in the post-test score means of both groups was found. The resulting effect size was 1.59 which indicates a strong effect. Furthermore, an increase in scores of at least 10.91% and at most 21% was found.

The null hypothesis was rejected and the alternative hypothesis was failed to reject. It was concluded that visualization as a supplement to teaching significantly increases the conceptual understanding and adaptive reasoning of high school calculus.

There is a dire need for mathematics educational practices to broaden the focus from Procedural Fluency and encompass Conceptual Understanding and Adaptive Reasoning as well. Training only for Procedural Fluency or computations leads to a tendency for memorization and mechanically implementing a set algorithm, with little to no awareness of the importance of different mathematical operations and symbols, their relationships and significance to the practical world. The use of visualization can deepen conceptual understanding by helping the students form deeper and meaningful connections between concepts, which in turn leads to an awareness of when and why these concepts are useful in the real life.

7.1. Future Directions

Mathematics education and learning is a field that has a lot of room for reforms based on research. The five strands of mathematical proficiency were formulated in 2001 and since then some research attempts have been conducted regarding their incorporation in the classroom, in assessment and on measuring their levels in students. However, one area that needs the most work is the fifth strand i.e. Productive Disposition.

(Siegfried, 2012, p. 199) in his doctoral dissertation states that productive disposition is a little-researched construct, and its connection to the other four strands of mathematical proficiency is completely unexplored. It is possible that this is because productive disposition embodies an attitude or outlook towards the subject, therefore incorporates a psychological aspect. Thus research methods and measurement for this strand might be different from the other four strands. However, given the interdependence of all five strands, it is very important for research work to be conducted on each of them.

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Appendices

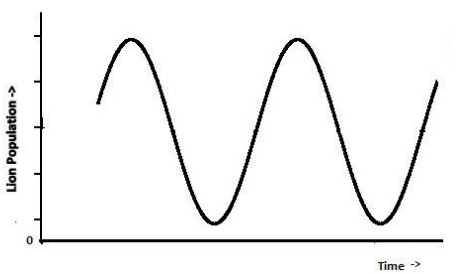
Appendix A

Pre-test

The pre-test questionnaire along with its marking rubrics is given below.

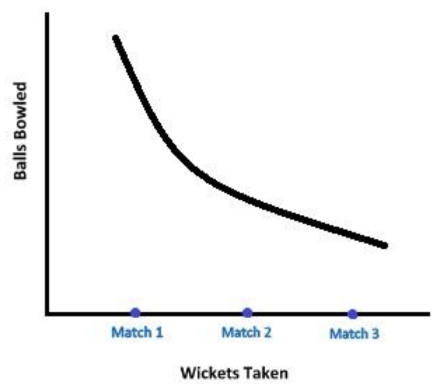
Questions	Rubrics
1. Mr. X is a Formula 1 car racer, which of the following cars should he choose? (Arbitrary Units used) a. Maximum Speed = 50, Maximum Acceleration = 5 b. Maximum Speed = 40, Maximum Acceleration = 10 [1]	I. b II. Higher Acceleration is beneficial for a racer as maximum speed can be attained quicker as compared to a car with a lower acceleration. [2 marks]

And Why? [1]	
2. The size of the population of lions and sheep in a jungle is related to each other (because a lion is a predator and feeds on sheep). *An increase in the lion population causes a decrease in the sheep population and vice versa. However, after a certain decrease in the sheep population, the reduced number of sheep available causes the lion population to start decreasing.* Assuming that the population changes for both animals are **periodic, do not have any abrupt changes with respect to time and that the population of either animal does not fall to zero at any time**, what sort of relationship will the population sizes have? Draw in the space provided. [2]	I. Two sinusoidal waves, inverted, with a slight phase shift. [2 marks] II. Two sinusoidal waves, inverted [1 mark] III. Two square waves/Any other waveform. [0 marks]

	
3. If the change in population size of lions is shown by the following graph, how can the change in population size of sheep be drawn? Draw on the graph below. [2] 	I. A corresponding inverted graph [1 mark] II. A corresponding inverted graph with a slight phase shift [2 marks] III. The same graph with a slight phase shift/Any other waveform. [0 marks]
4. What does the Speedometer of a car measure over a period of one hour? [1] a) Instantaneous Speed b) Instantaneous Velocity c) Average Speed d) Average Velocity	a.
5. Why is Displacement the Integral of Velocity? [2]	I. The integral is the antiderivative; differentiating (rate of change of)

	displacement gives velocity, therefore integrating velocity will give displacement. [2 marks] II. Integration gives area under the curve, displacement is the area under the velocity time graph. [1 mark] III. Any other answer [0 marks]
6. What will we get if we differentiate displacement twice? [1]	Acceleration
7. What will we get if we integrate the final result of the question above? [1]	Velocity
8. Global warming causes changes in the climate of our planet. If data about the maximum and minimum temperatures of a certain city over the past 50 years is given, how can a researcher observe to **what extent** the city been effected by global warming? [2]	I. Differentiating the temperatures over time gives the rate of change in temperature over 50 years (the extent to which the city has been effected by global warming). [2 marks] II. By calculating the difference in maximum and minimum temperatures [1 mark] III. Any other answer [0 marks]

9. If the data for the max and mini temperatures over the past 50 years is given for 10 different cities, how can a researcher observe which city has been **most effected** by global warming? [1]	I. The city with the highest value for differentiation has been the most effected [1 mark] II. Any other answer [0 marks]
10. The graph of the **bowling strike rate** of a bowler is plotted below. *(Bowling Strike Rate is defined as the average number of balls bowled per wicket taken*). What can you say about the performance of the baller? [1] a. His performance is improving b. His performance is deteriorating c. There is no change in his performance d. There is insufficient information to make a comment about his performance.	a.

Appendix B

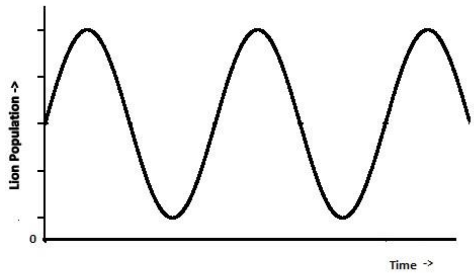
Post-test

The post-test questionnaire along with its marking rubrics is given below.

Questions	**Rubrics**

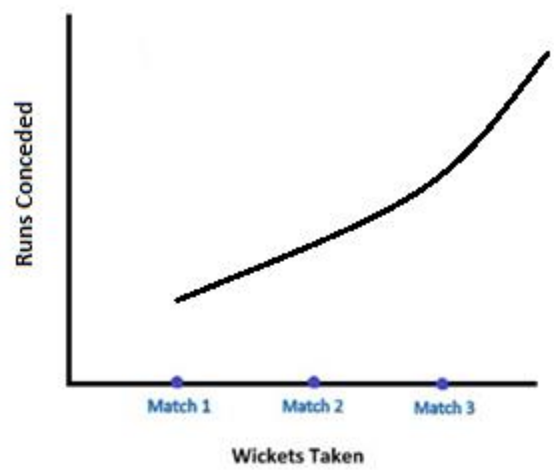
1. Anna wants to conduct a toys train race in her living room for her sons' friends on his 4th birthday party. She chooses two trains with the following specs: (Arbitrary Units used) a. Maximum Speed = 20, Maximum Acceleration = 5 b. Maximum Speed = 15, Maximum Acceleration = 10 Which toy train is more likely to win the race? And Why?	I. b II. Higher Acceleration ensures that maximum speed is attained quicker as compared to a train with a lower acceleration.
2. The size of the population of lion and sheep in a jungle is related to each other (because a lion is a predator and feeds on sheep). *A decrease in the lion population causes an increase in the sheep population and vice versa. However, after a certain increase in the sheep population, the increased number of sheep available causes the lion population to start increasing.*	I. Two sinusoidal waves, inverted. [1 mark] II. Two sinusoidal waves, inverted with a slight phase shift. [2 marks] III. Two square waves/Any other waveform. [0 marks]

Assuming that the population changes for both animals are **periodic, do not have any abrupt changes with respect to time and that the population of either animal does not fall to zero at any time**, what sort of relationship will the population sizes have? Draw in the space provided. 	
3. If the change in population size of lions is shown by the following graph, how can the change in population size of sheep be drawn?	I. A corresponding inverted graph [1 mark] II. A corresponding inverted graph with a slight phase shift [2 marks] III. The same graph with a slight phase shift/Any other waveform. [0 marks]

	
4. What does the Milometer/Odometer of a car measure? a. Distance Travelled b. Displacement Covered c. Average Speed d. Average Velocity	a.
5. Why is Velocity the Integral of Acceleration?	I. The integral is the antiderivative; differentiating (rate of change of) velocity gives acceleration, therefore integrating acceleration will give velocity. [2 marks] II. Integration gives area under the curve, velocity is the area under the acceleration time graph. [1

	mark] III. Any other answer [0 marks]
6. What will we get if we integrate acceleration twice?	Displacement
7. What will we get if we integrate the final result of the question above?	Velocity
8. Amount of water provided causes changes in the yield of crops (amount of crop harvested). If data about the maximum and minimum yield of a certain farm over the past 5 years is given, how can a researcher **observe to what extent** the yield been effected by amount of water provided?	I. Differentiating the yields over time gives the rate of change in yield over 5 years (the extent to which the yield has been effected by amount of water provided). [2 marks] II. By calculating the difference in maximum and minimum yield [1 mark] III. Any other answer [0 marks]
9. If the data for the maximum and minimum crop yields over the past 5 years is given for	I. The farm with the highest value for differentiation

10 different farms, how can a researcher observe which farm has been **most effected** by amount of water provided?	has been the most effected [1 mark] II. Any other answer [0 marks]
10. The graph of the **bowling average** of a bowler is plotted below. *(The average number of runs conceded per wicket*). What can you say about the performance of the baller? a. His performance is improving from Match 1 to Match 3. b. His performance is deteriorating from Match 1 to Match 3. c. There is no change in his performance from Match 1 to Match 3. d. There is insufficient information to make a comment about his performance.	b.


Runs Conceded
Match 1
Match 2
Match 3
Wickets Taken